

ZETA INTEGRALS AND INTEGRAL GEOMETRY IN THE SPACE OF RECTANGULAR MATRICES

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ABSTRACT. The paper is devoted to interrelation between the zeta distribution $\mathcal{Z}(f, \alpha - n) = \int f(x) \det(x'x)^{(\alpha-n)/2} dx$ and the Radon transform on the space $\mathfrak{M}_{n,m}$ of $n \times m$ real matrices $x = (x_{i,j})$. We present a self-contained proof of the Fourier transform formula for this distribution. Our method differs from that of J. Faraut and A. Korányi [FK] in the part related to justification of the corresponding Bernstein identity. We suggest a new proof of this identity based on explicit representation of the radial part of the Cayley-Laplace operator $\Delta = \det(\partial' \partial)$, $\partial = (\partial_{i,j})_{n \times m}$. We also study convolutions with normalized zeta distributions, and the corresponding Riesz potentials. The results are applied to investigation of Radon transforms on $\mathfrak{M}_{n,m}$.

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1. INTRODUCTION

Diverse problems of integral geometry in the N -dimensional real space \mathbb{R}^N get a new flavor if the dimension N has the form $N = nm$, and \mathbb{R}^N is identified with the space $\mathfrak{M}_{n,m}$ of $n \times m$ real matrices $x = (x_{i,j})$.

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If we accept this point of view, a number of new “higher rank” phenomena will come into play. In the present paper we focus on analytic tools which allow to investigate the properly defined Radon transform $f(x) \rightarrow \hat{f}(\tau)$ on $\mathfrak{M}_{n,m}$. This transform assigns to a function $f(x)$ on $\mathfrak{M}_{n,m}$ a collection of integrals of f over the so-called *matrix k -planes* τ in $\mathfrak{M}_{n,m}$. Each such plane is, in fact, an ordinary km -dimensional plane in \mathbb{R}^{nm} , but the manifold \mathfrak{T} of all matrix k -planes is essentially “smaller” than the manifold of all km -dimensional planes in \mathbb{R}^{nm} . The main problem is to reconstruct $f(x)$ from known data $\hat{f}(\tau)$, $\tau \in \mathfrak{T}$, for possibly large class of functions f .

In the rank-one case $m = 1$ this problem is well investigated; see, e.g., [E], [Hel], [Ru2], and references therein. The standard tools are the Fourier analysis, convolution operators, and the corresponding real-variable techniques, including approximation to the identity, singular integral operators, and fractional integrals. In the higher rank case $m \geq 2$, all these tools still require a further development.

Let us pass to details. Suppose that $f(x)$ is a Schwartz function on $\mathfrak{M}_{n,m}$, $n \geq m$. The Fourier transform of f is defined by

$$(1.1) \quad (\mathcal{F}f)(y) = \int_{\mathfrak{M}_{n,m}} \exp(\text{tr}(iy'x)) f(x) dx, \quad y = (y_{i,j}) \in \mathfrak{M}_{n,m},$$

where y' denotes the transpose of y , $dx = \prod_{i=1}^n \prod_{j=1}^m dx_{i,j}$. This is the ordinary Fourier transform on \mathbb{R}^{nm} . Given a complex number α , we denote

$$(1.2) \quad \mathcal{Z}(f, \alpha - n) = \int_{\mathfrak{M}_{n,m}} f(x) |x|_m^{\alpha-n} dx$$

where $|x|_m = \det(x'x)^{1/2}$ is the volume of the parallelepiped spanned by the column-vectors of the matrix x . For $m = 1$, $|x|_m$ is the usual euclidean norm on \mathbb{R}^n . Integrals like (1.2) are known as *zeta integrals*, the \mathcal{S}' -distribution defined by analytic continuation of (1.2) is called a *zeta distribution*, and the corresponding function $\alpha \rightarrow \mathcal{Z}(f, \alpha - n)$ is sometimes called a *zeta function*.

We begin our study by giving a relatively simple self-contained proof of the well-known functional relation

$$(1.3) \quad \frac{\mathcal{Z}(f, \alpha - n)}{\Gamma_m(\alpha/2)} = \pi^{-nm/2} 2^{m(\alpha-n)} \frac{\mathcal{Z}(\mathcal{F}f, -\alpha)}{\Gamma_m((n-\alpha)/2)},$$

where the normalizing denominators are Siegel gamma functions associated to the cone of positive definite $m \times m$ matrices. Both sides of

(1.3) are understood in the sense of analytic continuation. The equality (1.3) is of fundamental importance for the sequel. It is, in fact, a Fourier transform formula for the zeta distribution. Convolutions with zeta distributions coincide (up to normalization) with Riesz potentials on $\mathfrak{M}_{n,m}$ playing a vital role in integral geometry and analysis on matrix spaces [OR1], [P], [Sh].

Our interest to this topic is also motivated by the following. A number of proofs of the equality (1.3) and its generalizations can be found in the literature; see, e.g., [Cl], [Far], [FK], [Ge], [Ra], [Sh]. Unfortunately, these proofs are not “self-contained enough” and some important technical details are skipped. These details are crucial by taking into account an essential difference between the case $2m < n + 2$ and $2m \geq n + 2$. This difference is not indicated in some papers. The crux is that the distribution on the left-hand side of (1.3) is regular if and only if $\operatorname{Re} \alpha > m - 1$, whereas the right-hand side is regular if and only if $\operatorname{Re} \alpha < n - m + 1$. For $2m \geq n + 2$ these two sets on the complex plane are separated one from another! In the case $2m < n + 2$ the proof is elementary [OR1]. For $2m \geq n + 2$, justification of (1.3) represents a difficult problem.

The phenomenon of lack of common domain of regularity was investigated by E. Stein [St1] for Riesz distributions on the space of complex $n \times n$ matrices. For $n = m$, the formula (1.3) includes the result of S. S. Gelbart [Ge, Chapter IV] and a special case of Proposition II-9 of M. Raïs [Ra]. For $n \geq m$, this formula can be found in [Far], [FK, Theorem XVI, 4.3], and [Cl] in a more general context of representations of Jordan algebras; see also [Sh, p. 108]. For more information on zeta distributions, see [Ig], [SS], [Shin], and references therein.

Our proof of (1.3) utilizes the idea from [FK] to employ \mathcal{K} -Bessel functions and the so-called Bernstein identity

$$(1.4) \quad \Delta |x|_m^\lambda = \mathcal{B}(\lambda) |x|_m^{\lambda-2},$$

$$\mathcal{B}(\lambda) = (-1)^m \prod_{i=0}^{m-1} (\lambda + i)(2 - n - \lambda + i),$$

where Δ is the Cayley-Laplace operator defined by $\Delta = \det(\partial' \partial)$, $\partial = (\partial_{i,j})_{n \times m}$. This identity amounts to pioneering papers of J. Bernstein; see, e.g., [B], [Ig]. A novelty of our approach is that we first derive an explicit formula for the radial part of Δ , and then use it for justification of (1.4). It is worth noting that the proof of the Bernstein identity in [FK] and [Cl] employs the heat kernel and integration by parts over the corresponding cone. Justification of this integration by parts was unfortunately skipped. That was one of the reasons why we tried to

find an alternative proof. We hope that our method can be extended to more general settings similar to those in [FK] and [Cl].

The second concern of the paper is the so-called *Wallach set* \mathcal{W} of the normalized zeta distribution

$$(1.5) \quad (\zeta_\alpha, f) = a.c. \frac{1}{\Gamma_m(\alpha/2)} \int_{\mathfrak{M}_{n,m}} f(x) |x|_m^{\alpha-n} dx, \quad f \in \mathcal{S}(\mathfrak{M}_{n,m}),$$

where “a.c.” abbreviates analytic continuation. This is an entire function of α . The set \mathcal{W} consists of two components, one of which is continuous and another is discrete. The continuous component $\{\alpha : \operatorname{Re} \alpha > m - 1\}$ is constituted by those $\alpha \in \mathbb{C}$ for which the distribution (1.5) is regular, that is, the integral in (1.5) absolutely converges. The discrete component consists of non-negative integers $\alpha = k$, $k = 0, 1, \dots, m - 1$ (outside of the domain of regularity), for which the distribution (1.5) is a positive measure. We obtain explicit representation of ζ_α for all $\alpha \in \mathcal{W}$, and investigate convolutions $(\zeta_\alpha * f)(x)$ assuming $f \in L^p(\mathfrak{M}_{n,m})$. In particular, under some natural restrictions on n and m , we prove that these convolutions are well defined almost everywhere on $\mathfrak{M}_{n,m}$ provided

$$(1.6) \quad 1 \leq p < \frac{n}{\operatorname{Re} \alpha + m - 1}.$$

This result agrees with the case $m = 1$ in [St2]. The question whether the condition (1.6) is sharp remains open.

Another normalization of the integral (1.2) gives rise to the Riesz potential

$$(1.7) \quad (I^\alpha f)(x) = \frac{1}{\gamma_{n,m}(\alpha)} \int_{\mathfrak{M}_{n,m}} f(x - y) |y|_m^{\alpha-n} dy,$$

$$\gamma_{n,m}(\alpha) = \frac{2^{\alpha m} \pi^{nm/2} \Gamma_m(\alpha/2)}{\Gamma_m((n - \alpha)/2)}, \quad \alpha \neq n - m + 1, n - m + 2, \dots,$$

the Fourier transform of which is $|y|_m^{-\alpha} (\mathcal{F}f)(y)$ in a suitable sense. The operator I^α can be regarded as the $(-\alpha/2)$ th power of $(-1)^m \Delta$ where Δ is the Cayley-Laplace operator.

Our investigation of convolutions $(\zeta_\alpha * f)(x)$ and the Riesz potentials $I^\alpha f$ employs matrix modification of the Gauss-Weierstrass integral (see [SW], [Ta] for $m = 1$) defined by

$$(1.8) \quad (W_t f)(x) = \int_{\mathfrak{M}_{n,m}} h_t(x - y) f(y) dy,$$

where t is a positive definite $m \times m$ matrix, and

$$(1.9) \quad h_t(x) = (4\pi)^{-nm/2} \det(t)^{-n/2} \exp(-\operatorname{tr}(t^{-1}x'x)/4)$$

is the relevant heat kernel. This approach allows to avoid essential technical difficulties. The idea is to represent the Riesz potential through the lower-dimensional Gårding-Gindikin fractional integral by a simple formula

$$(1.10) \quad (W_t[I^\alpha f])(x) = (I_-^{\alpha/2} g_x)(t), \quad g_x(t) = (W_t f)(x),$$

provided $m - 1 < \operatorname{Re} \alpha < n - m + 1$. We recall that the Gårding-Gindikin fractional integral on the cone \mathcal{P}_m of positive definite matrices has the form

$$(1.11) \quad (I_-^\lambda g)(t) = \frac{1}{\Gamma_m(\lambda)} \int_t^\infty g(\tau) \det(\tau - t)^{\lambda-d} d\tau,$$

where $d = (m + 1)/2$, $\operatorname{Re} \lambda > d - 1$, and integration is performed over all $\tau \in \mathcal{P}_m$ so that $\tau - t \in \mathcal{P}_m$. These integrals were introduced by Lars Gårding [Gå1] who wrote [Gå2]:

“... Actually the origin of my integral was a statistics paper (samples of mean, variance etc. from a multivariate Gaussian distribution). The idea to use it in analysis came from my many meetings with my beloved teacher Marcel Riesz”.

Integrals (1.11) were substantially generalized by S. Gindikin [Gi]. They have proved to be an important tool in PDE [VG], [Rab], and in the theory of Radon transforms on Grassmann manifolds and matrix spaces [GR], [OR2], [Ru3]. If $t = 0$, then W_t turns into the identity operator and (1.10) reads

$$(1.12) \quad (I^\alpha f)(x) = \frac{1}{\Gamma_m(\alpha/2)} \int_{\mathcal{P}_m} |t|^{\alpha/2-d} (W_t f)(x) dt.$$

Our application of zeta distributions to integral geometry is based on intimate connection between the Riesz potentials (1.7) and the matrix k -plane Radon transform $f \rightarrow \hat{f}$ on $\mathfrak{M}_{n,m}$. This connection is realized via the generalized Fuglede formula

$$(1.13) \quad (\hat{f})^\vee(x) = \operatorname{const} (I^k f)(x)$$

which is well known for $m = 1$ [Fu], [Hel]. Here the left-hand side is the mean value of $\hat{f}(\tau)$ over all matrix k -planes τ “passing through x ”. For sufficiently good functions f , this formula was established in [OR1]. We justify it for all $f \in L^p(\mathfrak{M}_{n,m})$, $1 \leq p < n/(k + m - 1)$,

and apply to the inversion problem for the Radon transform $f \rightarrow \hat{f}$. Solution to this problem is presented in the framework of the relevant theory of distributions. For $m = 1$, such a theory was developed by V.I. Semyanistyi [Se].

The paper is organized as follows. In Section 2 we establish our notation and recall some basic facts important for the sequel. In Section 3 we derive a formula for the radial part of the Cayley-Laplace operator and prove the Bernstein identity (1.4). In Section 4 we study the zeta integral (1.2), prove (1.3), and obtain explicit representations of the normalized zeta distribution (1.5) in the discrete part of the Wallach set. Convolutions with zeta distributions and Riesz potentials of L^p functions are investigated in Section 5. Section 6 is devoted to applications of the results of preceding sections to integral geometry. We define the matrix k -plane Radon transform and justify the generalized Fuglede formula (1.13) for $f \in L^p(\mathfrak{M}_{n,m})$, $1 \leq p < n/(k + m - 1)$. Then we introduce the space Φ' of distributions of the Semyanistyi type. Two Φ' -distributions coincide if and only if their Fourier transforms differ by a tempered distribution supported by the singular set $\{y : y \in \mathfrak{M}_{n,m}, \text{rank}(y) < m\}$. An inversion formula for the Radon transform $f \rightarrow \hat{f}$ for $f \in L^p(\mathfrak{M}_{n,m})$ is then obtained in the Φ' -sense. For convenience of the reader, evaluation of some useful integrals is presented in Appendix.

The paper contains a number of open problems that might be of interest. These are stated in Remarks 5.5, 5.14, and 6.9.

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2. PRELIMINARIES

2.1. Matrix spaces. Let $\mathfrak{M}_{n,m}$ be the space of real matrices having n rows and m columns. We identify $\mathfrak{M}_{n,m}$ with the real Euclidean space \mathbb{R}^{nm} . The letters x, y, r, s , etc. stand for both the matrices and points since it is always clear from the context which is meant. If $x = (x_{i,j}) \in \mathfrak{M}_{n,m}$, we write $dx = \prod_{i=1}^n \prod_{j=1}^m dx_{i,j}$ for the elementary volume in $\mathfrak{M}_{n,m}$. In the following x' denotes the transpose of x , I_m is the identity $m \times m$ matrix, 0 stands for zero entries. Given a square matrix a , we denote by $|a|$ the absolute value of the determinant of a ; $\text{tr}(a)$ stands for the trace of a .

Let \mathcal{S}_m be the space of $m \times m$ real symmetric matrices $s = (s_{i,j})$, $s_{i,j} = s_{j,i}$. It is a measure space isomorphic to $\mathbb{R}^{m(m+1)/2}$ with the volume element $ds = \prod_{i \leq j} ds_{i,j}$. We denote by \mathcal{P}_m the open convex cone of positive definite matrices in \mathcal{S}_m ; $\bar{\mathcal{P}}_m$ is the closure of \mathcal{P}_m that consists of positive semi-definite matrices. For $r \in \mathcal{P}_m$ we write $r > 0$. The inequality $r_1 > r_2$ means $r_1 - r_2 \in \mathcal{P}_m$. If a and b are positive semi-definite matrices, the symbol $\int_a^b f(s)ds$ denotes integration over the set

$$\{s : s \in \mathcal{P}_m, a < s < b\} = \{s : s - a \in \mathcal{P}_m, b - s \in \mathcal{P}_m\}.$$

The group $G = GL(m, \mathbb{R})$ of real non-singular $m \times m$ matrices g acts on \mathcal{P}_m transitively by the rule $r \rightarrow g'rg$. The corresponding G -invariant measure on \mathcal{P}_m is

$$(2.1) \quad d_*r = |r|^{-d}dr, \quad |r| = \det(r), \quad d = (m+1)/2$$

[Te, p. 18]. The cone \mathcal{P}_m is a G -orbit in \mathcal{S}_m of the identity matrix I_m . The boundary $\partial\mathcal{P}_m$ of \mathcal{P}_m is a union of G -orbits of $m \times m$ matrices

$$e_k = \begin{bmatrix} I_k & 0 \\ 0 & 0 \end{bmatrix}, \quad k = 0, 1, \dots, m-1.$$

More information about the boundary structure of \mathcal{P}_m can be found in [FK, p. 72], and [Bar, p. 78].

We denote by T_m a group of upper triangular matrices

$$(2.2) \quad t = \begin{bmatrix} t_{1,1} & & & \\ & \cdot & t_* & \\ & & \cdot & \\ 0 & & & \cdot \\ & & & & t_{m,m} \end{bmatrix}, \quad t_{i,i} > 0,$$

$$t_* = \{t_{i,j} : i < j\} \in \mathbb{R}^{m(m-1)/2}.$$

Each $r \in \mathcal{P}_m$ has a unique representation $r = t't$, $t \in T_m$, so that

$$(2.3) \quad \begin{aligned} \int_{\mathcal{P}_m} f(r)dr &= \int_0^\infty t_{1,1}^m dt_{1,1} \int_0^\infty t_{2,2}^{m-1} dt_{2,2} \dots \\ &\times \int_0^\infty t_{m,m} \tilde{f}(t_{1,1}, \dots, t_{m,m}) dt_{m,m}, \end{aligned}$$

$$\tilde{f}(t_{1,1}, \dots, t_{m,m}) = 2^m \int_{\mathbb{R}^{m(m-1)/2}} f(t't) dt_*, \quad dt_* = \prod_{i < j} dt_{i,j},$$

[Te, p. 22] , [Mu, p. 592]. In the last integration, the diagonal entries of the matrix t are given by the arguments of \tilde{f} , and the strictly upper triangular entries of t are variables of integration.

Let us recall some useful formulas for Jacobians.

Lemma 2.1. (see, e.g., [Mu, pp. 57–59]).

(i) If $x = ayb$, where $y \in \mathfrak{M}_{n,m}$, $a \in GL(n, \mathbb{R})$, $b \in GL(m, \mathbb{R})$, then $dx = |a|^m |b|^n dy$.

(ii) If $r = q'sq$, where $s \in S_m$, $q \in GL(m, \mathbb{R})$, then $dr = |q|^{m+1} ds$.

(iii) If $r = s^{-1}$, where $s \in \mathcal{P}_m$, then $r \in \mathcal{P}_m$, and $dr = |s|^{-m-1} ds$.

In the following $(\lambda, m) = \lambda(\lambda+1) \dots (\lambda+m-1)$ is the Pochhammer symbol, δ_{ij} is the Kronecker delta; “a.c.” abbreviates analytic continuation. To facilitate presentation, we shall use the symbols “ \simeq ” and “ \lesssim ”, instead of “ $=$ ” and “ \leq ”, respectively, to indicate that the corresponding relation holds up to a constant multiple. All standard spaces of functions of matrix argument, say, $x = (x_{i,j})$, are identified with the corresponding spaces of functions of nm variables $x_{1,1}, x_{1,2}, \dots, x_{n,m}$. The Fourier transform of a function $f \in L^1(\mathfrak{M}_{n,m})$ is defined by

$$(2.4) \quad (\mathcal{F}f)(y) = \int_{\mathfrak{M}_{n,m}} \exp(\operatorname{tr}(iy'x)) f(x) dx, \quad y \in \mathfrak{M}_{n,m}.$$

2.2. Gamma functions, beta functions, and Bernstein polynomials. The Siegel gamma function associated to the cone \mathcal{P}_m is defined by

$$(2.5) \quad \Gamma_m(\alpha) = \int_{\mathcal{P}_m} \exp(-\operatorname{tr}(r)) |r|^{\alpha-d} dr, \quad d = (m+1)/2.$$

It is easy to check [Mu, p. 62], that the integral (2.5) converges absolutely if and only if $\operatorname{Re} \alpha > d - 1$, and represents a product of ordinary gamma functions:

$$(2.6) \quad \Gamma_m(\alpha) = \pi^{m(m-1)/4} \prod_{j=0}^{m-1} \Gamma(\alpha - j/2).$$

This implies

$$(2.7) \quad (-1)^m \frac{\Gamma_m(1 - \alpha/2)}{\Gamma_m(-\alpha/2)} = 2^{-m} \frac{\Gamma(\alpha + m)}{\Gamma(\alpha)} = 2^{-m}(\alpha, m),$$

$(\alpha, m) = \alpha(\alpha+1) \dots (\alpha+m-1)$ being the Pochhammer symbol. If $1 \leq k < m$, $k \in \mathbb{N}$, then

$$(2.8) \quad \Gamma_m(\alpha) = \pi^{k(m-k)/2} \Gamma_k(\alpha) \Gamma_{m-k}(\alpha - k/2),$$

$$(2.9) \quad \frac{\Gamma_m(\alpha)}{\Gamma_m(\alpha + k/2)} = \frac{\Gamma_k(\alpha + (k - m)/2)}{\Gamma_k(\alpha + k/2)}.$$

The beta function of the cone \mathcal{P}_m is defined by

$$(2.10) \quad B_m(\alpha, \beta) = \int_0^{I_m} |r|^{\alpha-d} |I_m - r|^{\beta-d} dr, \quad d = (m + 1)/2.$$

This integral converges absolutely if and only if $\operatorname{Re} \alpha, \operatorname{Re} \beta > d - 1$, and obeys the classical relation [FK, p. 130]

$$(2.11) \quad B_m(\alpha, \beta) = \frac{\Gamma_m(\alpha) \Gamma_m(\beta)}{\Gamma_m(\alpha + \beta)}.$$

Let $r = (r_{i,j}) \in \mathcal{P}_m$. We define the following differential operator acting in the r -variable:

$$(2.12) \quad D \equiv D_r = \det \left(\eta_{i,j} \frac{\partial}{\partial r_{i,j}} \right), \quad \eta_{i,j} = \begin{cases} 1 & \text{if } i = j \\ 1/2 & \text{if } i \neq j. \end{cases}$$

Lemma 2.2.

(i) For $r \in \mathcal{P}_m$ and $z \in \mathcal{S}_m^{\mathbb{C}}$ (the complexification of \mathcal{S}_m),

$$(2.13) \quad D_r[\exp(-\operatorname{tr}(rz))] = (-1)^m \det(z) \exp(-\operatorname{tr}(rz)).$$

(ii) For $r \in \mathcal{P}_m$ and $\alpha \in \mathbb{C}$,

$$(2.14) \quad D(|r|^{\alpha-d} / \Gamma_m(\alpha)) = |r|^{\alpha-1-d} / \Gamma_m(\alpha - 1), \quad d = (m + 1)/2.$$

The proof of this statement is simple and can be found in [Gå1, p. 813], [FK, p. 125].

By changing notation and using (2.7), one can write (2.14) as

$$(2.15) \quad D|r|^{\alpha} = b(\alpha)|r|^{\alpha-1},$$

where

$$(2.16) \quad b(\alpha) = \alpha(\alpha + 1/2) \cdots (\alpha + d - 1)$$

is the so-called *Bernstein polynomial* of the determinant [FK].

2.3. Bessel functions of matrix argument. We recall some facts from [Herz] and [FK].

2.3.1. *\mathcal{J} -Bessel functions.* The \mathcal{J} -Bessel function $\mathcal{J}_\nu(r)$, $r \in \mathcal{P}_m$, can be defined in terms of the Laplace transform by the property

$$(2.17) \quad \int_{\mathcal{P}_m} \exp(-\operatorname{tr}(zr)) \mathcal{J}_\nu(r) |r|^{\nu-d} dr = \Gamma_m(\nu) \exp(-\operatorname{tr}(z^{-1})) \det(z)^{-\nu},$$

$$d = (m+1)/2, \quad z \in \mathcal{T}_m = \mathcal{P}_m + i\mathcal{S}_m$$

(the branch of the multi-valued function $\det(z)^{-\nu}$ is chosen so that for $z \in \mathcal{P}_m$, the argument of $\det(z)^{-\nu}$ is zero). This gives

$$(2.18) \quad \mathcal{J}_\nu(r) |r|^{\nu-d} = \frac{\Gamma_m(\nu)}{(2\pi i)^N} \int_{\operatorname{Re} z = \sigma_0} \exp(\operatorname{tr}(rz - z^{-1})) \det(z)^{-\nu} dz,$$

$\sigma_0 \in \mathcal{P}_m$, $N = m(m+1)/2$, or, by changing variable,

$$(2.19) \quad \mathcal{J}_\nu(r) = \frac{\Gamma_m(\nu)}{(2\pi i)^N} \int_{\operatorname{Re} z = \sigma_0} \exp(\operatorname{tr}(z - rz^{-1})) \det(z)^{-\nu} dz.$$

Both integrals are absolutely convergent for $\operatorname{Re} \nu > m$. Applying the operator (2.12), we obtain the following recurrence formulae:

$$(2.20) \quad D[\mathcal{J}_\nu(r) |r|^{\nu-d}] = \frac{\Gamma_m(\nu)}{\Gamma_m(\nu-1)} \mathcal{J}_{\nu-1}(r) |r|^{\nu-1-d},$$

$$(2.21) \quad D[\mathcal{J}_\nu(r)] = (-1)^m \frac{\Gamma_m(\nu)}{\Gamma_m(\nu+1)} \mathcal{J}_{\nu+1}(r).$$

The equation (2.20) allows to extend $\mathcal{J}_\nu(r)$ analytically to all $\nu \in \mathbb{C}$.

There is an intimate connection between the Fourier transform and \mathcal{J} -Bessel functions.

Theorem 2.3. [Herz, p. 492], [FK, p. 355] *Let $f(x)$ be an integrable function on $\mathfrak{M}_{n,m}$ of the form $f(x) = f_0(x'x)$, where f_0 is a function on \mathcal{P}_m . Then*

$$(2.22) \quad (\mathcal{F}f)(y) = \int_{\mathfrak{M}_{n,m}} \exp(\operatorname{tr}(iy'x)) f_0(x'x) dx = \frac{\pi^{nm/2}}{\Gamma_m(n/2)} \tilde{f}_0\left(\frac{y'y}{4}\right),$$

where

$$(2.23) \quad \tilde{f}_0(s) = \int_{\mathcal{P}_m} \mathcal{J}_{n/2}(rs) |r|^{n/2-d} f_0(r) dr, \quad d = (m+1)/2,$$

is the Hankel transform of f_0 .

This statement is a matrix generalization of the classical result of S. Bochner, see, e.g., [SW], Chapter IV, Theorem 3.3.

One can meet another notation for the \mathcal{J} -Bessel function in the literature. We follow the notation from [FK]. It relates to the notation $A_\delta(r)$ in [Herz] by the formula $\mathcal{J}_\nu(r) = \Gamma_m(\nu) A_\delta(r)$, $\delta = \nu - d$. For $m = 1$, the classical Bessel function $J_\nu(r)$ expresses through $\mathcal{J}_\nu(r)$ as

$$J_\nu(r) = \frac{1}{\Gamma(\nu + 1)} \left(\frac{r}{2}\right)^\nu \mathcal{J}_{\nu+1}\left(\frac{r^2}{4}\right).$$

2.3.2. \mathcal{K} -Bessel functions. Let, as above, $r \in \mathcal{P}_m$, $d = (m + 1)/2$. The \mathcal{K} -Bessel function $\mathcal{K}_\nu(r)$ is defined by

$$(2.24) \quad \mathcal{K}_\nu(r) = \int_{\mathcal{P}_m} \exp(\operatorname{tr}(-s - rs^{-1})) |s|^{\nu-d} ds,$$

or (replace s by s^{-1})

$$(2.25) \quad \mathcal{K}_\nu(r) = \int_{\mathcal{P}_m} \exp(\operatorname{tr}(-s^{-1} - rs)) |s|^{-\nu-d} ds,$$

see [Te], [FK]. For $m = 1$,

$$(2.26) \quad \mathcal{K}_\nu(r) = \int_0^\infty \exp(-s - r/s) s^{\nu-1} ds = 2r^{\nu/2} K_\nu(2\sqrt{r}),$$

K_ν being the classical Macdonald function.

Lemma 2.4. *Let $r \in \mathcal{P}_m$, $d = (m + 1)/2$.*

(i) *The integral (2.24) (or (2.25)) converges absolutely for all $\nu \in \mathbb{C}$, and is an entire function of ν .*

(ii) *The following estimates hold:*

(a) *For $\operatorname{Re} \nu > d - 1$:*

$$(2.27) \quad |\mathcal{K}_\nu(r)| \leq \Gamma_m(\operatorname{Re} \nu);$$

(b) *For $\operatorname{Re} \nu < 1 - d$:*

$$(2.28) \quad |\mathcal{K}_\nu(r)| \leq \Gamma_m(-\operatorname{Re} \nu) |r|^{\operatorname{Re} \nu};$$

(c) *For $1 - d \leq \operatorname{Re} \nu \leq d - 1$:*

$$(2.29) \quad |\mathcal{K}_\nu(r)| \leq \Gamma_m(d) + |r|^{1-d-\varepsilon} \Gamma_m(d - 1 + \varepsilon), \quad \forall \varepsilon > 0.$$

(iii) *If $\operatorname{Re} \nu < 1 - d$, then*

$$(2.30) \quad \lim_{\varepsilon \rightarrow 0} \varepsilon^{-m\nu} \mathcal{K}_\nu(\varepsilon r) = \Gamma_m(-\nu) |r|^\nu.$$

Proof. All statements, with probable exception of (2.29), are known [FK]. The estimate (2.27) follows from (2.24); (2.28) and (2.30) are consequences of (2.25). To prove (2.29), we write

$$\mathcal{K}_\nu(r) = \left(\int_{|s|>1} + \int_{|s|<1} \right) \exp(\operatorname{tr}(-s - rs^{-1})) |s|^{\nu-d} ds = I_1 + I_2.$$

For I_1 we have

$$|I_1| < \int_{|s|>1} \exp(\operatorname{tr}(-s)) ds < \Gamma_m(d)$$

provided $\operatorname{Re} \nu \leq d$. For I_2 , by changing variable $s \rightarrow s^{-1}$, we obtain

$$I_2 = \int_{|s|>1} \exp(\operatorname{tr}(-s^{-1} - rs)) |s|^{-\nu-d} ds.$$

If $\operatorname{Re} \nu \geq 1 - d$, then $|s|^{-\operatorname{Re} \nu - d} \leq |s|^{\varepsilon-1} \forall \varepsilon > 0$, and

$$\begin{aligned} |I_2| &\leq \int_{|s|>1} \exp(\operatorname{tr}(-s^{-1} - rs)) |s|^{\varepsilon-1} ds \\ &< \int_{\mathcal{P}_m} \exp(\operatorname{tr}(-rs)) |s|^{\varepsilon-1} ds \\ &= \Gamma_m(d - 1 + \varepsilon) |r|^{1-d-\varepsilon}. \end{aligned}$$

This gives (2.29). \square

2.4. Stiefel manifolds. For $n \geq m$, let $V_{n,m} = \{v \in \mathfrak{M}_{n,m} : v'v = I_m\}$ be the Stiefel manifold of orthonormal m -frames in \mathbb{R}^n . If $n = m$, then $V_{n,n} = O(n)$ is the orthogonal group in \mathbb{R}^n . The group $O(n)$ acts on $V_{n,m}$ transitively by the rule $g : v \rightarrow gv$, $g \in O(n)$, in the sense of matrix multiplication. The same is true for the special orthogonal group $SO(n)$ provided $n > m$. We fix the corresponding invariant measure dv on $V_{n,m}$ normalized by

$$(2.31) \quad \sigma_{n,m} \equiv \int_{V_{n,m}} dv = \frac{2^m \pi^{nm/2}}{\Gamma_m(n/2)}.$$

Lemma 2.5. [Mu, p. 589] *Let x and y be real matrices such that x is $n \times k$ and y is $m \times k$, $n \geq m$. Then $x'x = y'y$ if and only if there exists $v \in V_{n,m}$ such that $x = vy$. In particular, if x is $n \times m$, $n \geq m$, then there exists $v \in V_{n,m}$ such that $x = vy$, where $y = (x'x)^{1/2}$.*

Lemma 2.6. (polar decomposition). *Let $x \in \mathfrak{M}_{n,m}$, $n \geq m$. If $\text{rank}(x) = m$, then*

$$x = vr^{1/2}, \quad v \in V_{n,m}, \quad r = x'x \in \mathcal{P}_m,$$

and $dx = 2^{-m}|r|^{(n-m-1)/2}drdv$.

This statement and its generalizations can be found in different places, see, e.g., [Herz, p. 482], [Mu, pp. 66, 591], [FT, p. 130]. A modification of Lemma 2.6 in terms of upper triangular matrices $t \in T_m$ (see (2.2)) reads as follows.

Lemma 2.7. *Let $x \in \mathfrak{M}_{n,m}$, $n \geq m$. If $\text{rank}(x) = m$, then*

$$x = vt, \quad v \in V_{n,m}, \quad t \in T_m,$$

so that

$$dx = \prod_{j=1}^m t_{j,j}^{n-j} dt_{j,j} dt_* dv, \quad dt_* = \prod_{i < j} dt_{i,j}.$$

Proof. This statement is also well known. It can be easily derived from Lemma 2.6 and (2.3). Indeed, if $\text{rank}(x) = m$, then $x'x \in \mathcal{P}_m$ and there exists $t \in T_m$ such that $x'x = t't$. We set $v = xt^{-1}$. Then $v'v = I_m$ and, therefore, $v \in V_{n,m}$. This proves the representation $x = vt$, where $v \in V_{n,m}$ and $t \in T_m$. Furthermore, by Lemma 2.6 and (2.3),

$$\begin{aligned} \int_{\mathfrak{M}_{n,m}} f(x) dx &= 2^{-m} \int_{V_{n,m}} dv \int_{\mathcal{P}_m} |r|^{(n-m-1)/2} f(vr^{1/2}) dr \\ &\stackrel{(2.3)}{=} \int_{V_{n,m}} dv \int_{T_m} f(v(t't)^{1/2}) \prod_{j=1}^m t_{j,j}^{n-j} dt_{j,j} dt_* dv. \end{aligned}$$

Now we denote $\lambda = (t't)^{1/2}t^{-1} \in O(m)$, then change the order of integration, and set $v\lambda = u$. This gives

$$\int_{\mathfrak{M}_{n,m}} f(x) dx = \int_{T_m} \prod_{j=1}^m t_{j,j}^{n-j} dt_{j,j} dt_* \int_{V_{n,m}} f(ut) du,$$

and we are done. \square

3. RADIAL FUNCTIONS AND THE CAYLEY-LAPLACE OPERATOR

A function $f(x)$ on $\mathfrak{M}_{n,m}$ is called radial, if there exists a function $f_0(r)$ on \mathcal{P}_m such that $f(x) = f_0(x'x)$ for all (or almost all) matrices $x \in \mathfrak{M}_{n,m}$. One can readily check that f is radial if and only if it is $O(n)$ left-invariant, i.e., $f(\gamma x) = f(x)$ for all $\gamma \in O(n)$.

The *Cayley-Laplace operator* Δ on the space $\mathfrak{M}_{n,m}$ of matrices $x = (x_{i,j})$ is defined by

$$(3.1) \quad \Delta = \det(\partial' \partial).$$

Here ∂ is an $n \times m$ matrix whose entries are partial derivatives $\partial/\partial x_{i,j}$. In the Fourier transform terms, the action of Δ represents a multiplication by the polynomial $(-1)^m P(y)$, where $y = (y_{i,j}) \in \mathfrak{M}_{n,m}$,

$$P(y) = |y'y| = \det \begin{bmatrix} y_1 \cdot y_1 & \cdot & \cdot & y_1 \cdot y_m \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ y_m \cdot y_1 & \cdot & \cdot & y_m \cdot y_m \end{bmatrix},$$

y_1, \dots, y_m are column-vectors of the matrix y , and “ \cdot ” stands for the usual inner product in \mathbb{R}^n . Clearly, $P(y)$ is a homogeneous polynomial of degree $2m$ of nm variables $y_{i,j}$, and Δ is a homogeneous differential operator of order $2m$. For $m = 1$, it coincides with the Laplace operator on \mathbb{R}^n .

The Cayley-Laplace operator (3.1) and its generalizations were studied by S.P. Khokalo [Kh]. For $m > 1$, the operator Δ is not elliptic because $P(y) = 0$ for all non-zero matrices y of rank $< m$. Moreover, Δ is not hyperbolic, although, for some n, m and ℓ , its power Δ^ℓ enjoys the strengthened Huygens' principle; see [Kh] for details.

Our nearest goal is to find a radial part of Δ corresponding to the polar decomposition $x = vr^{1/2}$, $v \in V_{n,m}$, $r = x'x \in \mathcal{P}_m$. For $m = 1$, the classical result states that the radial part of the Laplace operator on \mathbb{R}^n is

$$L = \rho^{1-n} \frac{\partial}{\partial \rho} \rho^{n-1} \frac{\partial}{\partial \rho}, \quad \rho = |x|, \quad x \in \mathbb{R}^n.$$

By changing variable $r = \rho^2$, we get

$$(3.2) \quad L = 4r^{1-n/2} \frac{\partial}{\partial r} r^{n/2} \frac{\partial}{\partial r}.$$

The following statement is one of the main results of the paper. It extends (3.2) to the higher rank case.

Theorem 3.1. *Let $\Omega \subset \mathfrak{M}_{n,m}$ be an open set consisting of matrices of rank m ; $n \geq m \geq 1$. If $f(x) = f_0(x'x)$, $f_0(r) \in C^{2m}(\mathcal{P}_m)$, then for $x \in \Omega$,*

$$(3.3) \quad (\Delta f)(x) = (L f_0)(x'x),$$

where

$$(3.4) \quad L = 4^m |r|^{d-n/2} D |r|^{n/2-d+1} D, \quad d = (m+1)/2,$$

$D = \det[((1 + \delta_{ij})/2)\partial/\partial r_{i,j}]$ being the operator (2.12).

Proof. For $m = 1$, (3.4) coincides with (3.2). To prove the theorem, we first note that without loss of generality, one can assume f to be compactly supported away from the surface $\{x : \det(x'x) = 0\}$. Otherwise, $f(x)$ can be replaced by $f_1(x) = \varphi(x)\psi(x)f(x)$, where φ and ψ are radial cut-off functions of the form

$$\begin{aligned} \varphi(x) &= \varphi_0(\text{tr}(x'x)), & \psi(x) &= \psi_0(\det(x'x)), & \varphi_0, \psi_0 &\in C^\infty(\mathbb{R}_+), \\ \varphi_0(\rho) &= \begin{cases} 1, & \text{if } 0 \leq \rho < N, \\ 0, & \text{if } \rho \geq N+1, \end{cases} & \psi_0(\rho) &= \begin{cases} 0, & \text{if } 0 \leq \rho \leq \varepsilon, \\ 1, & \text{if } \rho \geq 2\varepsilon. \end{cases} \end{aligned}$$

The positive numbers N and ε should be chosen sufficiently large and small, respectively.

By the generalized Bochner formula (2.22),

$$\mathcal{F}[\Delta f](y) = (-1)^m |y'y|(\mathcal{F}f)(y) = \frac{\pi^{nm/2}}{\Gamma_m(n/2)} h\left(\frac{y'y}{4}\right),$$

where

$$h(s) = (-4)^m \int_{\mathcal{P}_m} |rs| \mathcal{J}_{n/2}(rs) |r|^{n/2-d-1} f_0(r) dr.$$

Let us transform $|rs| \mathcal{J}_{n/2}(rs)$. By (2.20) and (2.21),

$$\begin{aligned} D[|r|^{n/2+1-d} D \mathcal{J}_{n/2}(r)] &= (-1)^m \frac{\Gamma_m(n/2)}{\Gamma_m(n/2+1)} D[\mathcal{J}_{n/2+1}(r) |r|^{n/2+1-d}] \\ &= (-1)^m \mathcal{J}_{n/2}(r) |r|^{n/2-d}. \end{aligned}$$

Hence,

$$|r| \mathcal{J}_{n/2}(r) = (-1)^m (\tilde{L} \mathcal{J}_{n/2})(r), \quad \tilde{L} = (|r|^{d+1-n/2} D |r|^{n/2-d}) (|r| D).$$

Since $|r|^{d+1-n/2} D |r|^{n/2-d}$ and $|r| D$ are invariant differential operators with respect to the transformation $r \rightarrow grg'$, $g \in GL(m, \mathbb{R})$, [FK, p. 294], then so is \tilde{L} . Thus for any $s \in \mathcal{P}_m$,

$$|rs| \mathcal{J}_{n/2}(rs) = (-1)^m (\tilde{L} \mathcal{J}_{n/2})(sr) = (-1)^m \tilde{L}_r [\mathcal{J}_{n/2}(sr)],$$

where \tilde{L}_r stands for the operator \tilde{L} acting in the r -variable (here we use the symmetry property $\mathcal{J}_\nu(rs) = \mathcal{J}_\nu(s^{1/2} r s^{1/2})$). It follows that

$$\begin{aligned} h(s) &= 4^m \int_{\mathcal{P}_m} \tilde{L}_r [\mathcal{J}_{n/2}(sr)] |r|^{n/2-d-1} f_0(r) dr \\ &= 4^m \int_{\mathcal{P}_m} (D_r |r|^{n/2-d+1} D_r) [\mathcal{J}_{n/2}(sr)] f_0(r) dr. \end{aligned}$$

Owing to remark at the beginning of the proof, one can integrate by parts and get

$$\begin{aligned} h(s) &= (-4)^m \int_{\mathcal{P}_m} D_r[\mathcal{J}_{n/2}(sr)] |r|^{n/2-d+1} (Df_0)(r) dr \\ &= \int_{\mathcal{P}_m} \mathcal{J}_{n/2}(sr) |r|^{n/2-d} (Lf_0)(r) dr, \end{aligned}$$

where $L = 4^m |r|^{d-n/2} D |r|^{n/2-d+1} D$. Thus

$$\mathcal{F}[\Delta f](y) = \frac{\pi^{nm/2}}{\Gamma_m(n/2)} \int_{\mathcal{P}_m} \mathcal{J}_{n/2} \left(\frac{1}{4} r y' y \right) |r|^{n/2-d} (Lf_0)(r) dr,$$

which implies (3.3) . □

Example 3.2. Let $f(x) = |x|_m^\lambda$, $|x|_m = \det(x'x)^{1/2}$. By Theorem 3.1 and (2.15), $(\Delta f)(x) = \varphi(x'x)$, where

$$\begin{aligned} \varphi(r) &= 4^m |r|^{d-n/2} D |r|^{n/2-d+1} D |r|^{\lambda/2} \\ &= 4^m b(\lambda/2) |r|^{d-n/2} D |r|^{(n+\lambda)/2-d} \\ &= 4^m b(\lambda/2) b((n+\lambda)/2-d) |r|^{\lambda/2-1}. \end{aligned}$$

Thus, we have arrived at the following identity of the Bernstein type

$$(3.5) \quad \Delta |x|_m^\lambda = \mathcal{B}(\lambda) |x|_m^{\lambda-2},$$

where, owing to (2.16), the polynomial $\mathcal{B}(\lambda)$ has the form

$$(3.6) \quad \mathcal{B}(\lambda) = (-1)^m \prod_{i=0}^{m-1} (\lambda + i)(2 - n - \lambda + i).$$

An obvious consequence of (3.5) in a slightly different notation reads

$$(3.7) \quad \Delta^k |x|_m^{\alpha+2k-n} = B_k(\alpha) |x|_m^{\alpha-n},$$

$$\begin{aligned} (3.8) \quad B_k(\alpha) &= \prod_{i=0}^{m-1} \prod_{j=0}^{k-1} (\alpha - i + 2j)(\alpha - n + 2 + 2j + i) \\ &= B_k(n - \alpha - 2k). \end{aligned}$$

4. ZETA INTEGRALS

4.1. **Definition and example.** Let us consider the zeta integral

$$(4.1) \quad \mathcal{Z}(f, \alpha - n) = \int_{\mathfrak{M}_{n,m}} f(x) |x|_m^{\alpha-n} dx$$

where $f(x)$ is a Schwartz function on $\mathfrak{M}_{n,m}$, $n \geq m$, $|x|_m = \det(x'x)^{1/2}$. The following example gives a flavor of basic properties of $\mathcal{Z}(f, \alpha - n)$.

Example 4.1. Let $e(x) = \exp(-\text{tr}(x'x))$ be the Gaussian function. By Lemma 2.6, for $\text{Re } \alpha > m - 1$ we have

$$(4.2) \quad \begin{aligned} \mathcal{Z}(e, \alpha - n) &= 2^{-m} \sigma_{n,m} \int_{\mathcal{P}_m} |r|^{\alpha/2-d} \exp(-\text{tr}(r)) dr \\ &= c_{n,m} \Gamma_m(\alpha/2), \quad c_{n,m} = \frac{\pi^{nm/2}}{\Gamma_m(n/2)}. \end{aligned}$$

$d = (m + 1)/2$. On the other hand, the well known formula for the Fourier transform yields

$$(4.3) \quad (\mathcal{F}e)(y) = \pi^{nm/2} e(y/2),$$

and for $\text{Re } \alpha < n - m + 1$ we obtain

$$(4.4) \quad \begin{aligned} \mathcal{Z}(\mathcal{F}e, -\alpha) &= c_{n,m} \pi^{nm/2} \int_{\mathcal{P}_m} |r|^{(n-\alpha)/2-d} \exp(-\text{tr}(r/4)) dr \\ &= d_{n,m} \Gamma_m((n-\alpha)/2), \quad d_{n,m} = c_{n,m} \pi^{nm/2} 2^{m(n-\alpha)}. \end{aligned}$$

Thus, after analytic continuation we obtain the following meromorphic functions:

$$(4.5) \quad \mathcal{Z}(e, \alpha - n) = c_{n,m} \Gamma_m(\alpha/2), \quad \alpha \neq m - 1, m - 2, \dots,$$

$$(4.6) \quad \mathcal{Z}(\mathcal{F}e, -\alpha) = d_{n,m} \Gamma_m((n-\alpha)/2), \quad \alpha \neq n - m + 1, n - m + 2, \dots$$

These equalities imply the functional relation

$$(4.7) \quad \frac{\mathcal{Z}(e, \alpha - n)}{\Gamma_m(\alpha/2)} = \pi^{-nm/2} 2^{m(\alpha-n)} \frac{\mathcal{Z}(\mathcal{F}e, -\alpha)}{\Gamma_m((n-\alpha)/2)},$$

which is a prototype of similar formulas for much more general zeta functions. Since each side of (4.7) equals $c_{n,m}$, this formula extends to all complex α . Note that excluded values of α in (4.5) and (4.6) correspond to $m \geq 2$. If $m = 1$, they proceed with step 2, namely, $\alpha \neq 0, -2, -4, \dots$, and $\alpha \neq n, n + 2, \dots$, respectively.

In the following, throughout the paper, we assume $m \geq 2$.

Lemma 4.2. *Let $f \in \mathcal{S}(\mathfrak{M}_{n,m})$. For $\operatorname{Re} \alpha > m - 1$, the integral (4.1) is absolutely convergent, and for $\operatorname{Re} \alpha \leq m - 1$, it extends as a meromorphic function of α with the only poles $m - 1, m - 2, \dots$. These poles and their orders are exactly the same as of the gamma function $\Gamma_m(\alpha/2)$. The normalized zeta integral $\mathcal{Z}(f, \alpha - n)/\Gamma_m(\alpha/2)$ is an entire function of α .*

Proof. This statement is known; see, e.g., [Kh], [Sh]. We present the proof for the sake of completeness. The equality (4.5) says that the function $\alpha \rightarrow \mathcal{Z}(f, \alpha - n)$ has poles at least at the same points and at least of the same order as the gamma function $\Gamma_m(\alpha/2)$. Our aim is to show that no other poles occur, and the orders cannot exceed those of $\Gamma_m(\alpha/2)$. Let us transform (4.1) by passing to upper triangular matrices $t \in T_m$ according to Lemma 2.7. We have

$$(4.8) \quad \mathcal{Z}(f, \alpha - n) = \int_{\mathbb{R}_+^m} F(t_{1,1}, \dots, t_{m,m}) \prod_{i=1}^m t_{i,i}^{\alpha-i} dt_{i,i},$$

$$F(t_{1,1}, \dots, t_{m,m}) = \int_{\mathbb{R}^{m(m-1)/2}} dt_* \int_{V_{n,m}} f(vt) dv, \quad dt_* = \prod_{i < j} dt_{i,j}.$$

Since F extends as an even Schwartz function in each argument, it can be written as

$$F(t_{1,1}, \dots, t_{m,m}) = F_0(t_{1,1}^2, \dots, t_{m,m}^2),$$

where $F_0 \in \mathcal{S}(\mathbb{R}^m)$ (use, e.g., Lemma 5.4 from [Tr, p. 56]. Replacing $t_{i,i}^2$ by $s_{i,i}$, we represent (4.8) as a direct product of one-dimensional distributions

$$(4.9) \quad \mathcal{Z}(f, \alpha - n) = 2^{-m} \left(\prod_{i=1}^m (s_{i,i})_+^{(\alpha-i-1)/2}, F_0(s_{1,1}, \dots, s_{m,m}) \right),$$

which is a meromorphic function of α with the poles $m - 1, m - 2, \dots$, see [GSh1]. These poles and their orders coincide with those of the gamma function $\Gamma_m(\alpha/2)$, cf. (4.5). To normalize the function (4.9), following [GSh1], we divide it by the product

$$\prod_{i=1}^m \Gamma((\alpha - i + 1)/2) = \prod_{i=0}^{m-1} \Gamma((\alpha - i)/2) = \Gamma_m(\alpha/2) / \pi^{m(m-1)/4}.$$

As a result we obtain an entire function. □

4.2. A functional equation for the zeta integral. Let us prove another main result of the paper.

Theorem 4.3. *If $f \in \mathcal{S}(\mathfrak{M}_{n,m})$, $n \geq m$, then*

$$(4.10) \quad \frac{\mathcal{Z}(f, \alpha - n)}{\Gamma_m(\alpha/2)} = \pi^{-nm/2} 2^{m(\alpha-n)} \frac{\mathcal{Z}(\mathcal{F}f, -\alpha)}{\Gamma_m((n-\alpha)/2)}.$$

Proof. We recall that both sides of this equality are understood in the sense of analytic continuation. Furthermore, the distribution on the left-hand side of (4.10) is regular if and only if $\operatorname{Re} \alpha > m - 1$, whereas the right-hand side is regular if and only if $\operatorname{Re} \alpha < n - m + 1$. For $2m \geq n + 2$ these two sets have no common points. We consider the cases $2m < n + 2$ and $2m \geq n + 2$ separately.

(i) **The case $2m < n + 2$.** We start with the following equality:

$$(4.11) \quad \begin{aligned} & \int (\mathcal{F}f)(y) |\varepsilon I_m + y'y|^{-\alpha/2} dy \\ &= \frac{\mathfrak{M}_{n,m} \pi^{nm/2} \varepsilon^{m(n-\alpha)/2}}{\Gamma_m(\alpha/2)} \int_{\mathfrak{M}_{n,m}} f(x) \mathcal{K}_{(\alpha-n)/2} \left(\frac{\varepsilon}{4} x'x \right) dx, \quad \varepsilon > 0. \end{aligned}$$

Here $\mathcal{K}_{(\alpha-n)/2}$ is the \mathcal{K} -Bessel function (see Section 2.3.2), and

$$(4.12) \quad m - 1 < \operatorname{Re} \alpha < n - m + 1$$

(since $2m < n + 2$, the domain (4.12) is not void). Suppose for a moment, that (4.11) is true. Then the result follows if we pass to the limit as $\varepsilon \rightarrow 0$. Indeed, by (2.28) for all $\varepsilon > 0$ we have

$$(4.13) \quad |\varepsilon^{m(n-\alpha)/2} \mathcal{K}_{(\alpha-n)/2} \left(\frac{\varepsilon}{4} x'x \right)| \leq \Gamma_m \left(\frac{n - \operatorname{Re} \alpha}{2} \right) \left| \frac{x'x}{4} \right|^{(\operatorname{Re} \alpha - n)/2}.$$

Furthermore, by (2.30),

$$(4.14) \quad \lim_{\varepsilon \rightarrow 0} \varepsilon^{m(n-\alpha)/2} \mathcal{K}_{(\alpha-n)/2} \left(\frac{\varepsilon}{4} x'x \right) = 2^{m(n-\alpha)} \Gamma_m \left(\frac{n - \alpha}{2} \right) |x|_m^{\alpha-n}.$$

Hence, by the Lebesgue theorem on dominated convergence, we are done.

To prove (4.11), let $e_s(x) = \exp(-\operatorname{tr}(xsx')/4\pi)$, $s \in \mathcal{P}_m$. By the Plancherel formula,

$$(4.15) \quad |s|^{-n/2} \int_{\mathfrak{M}_{n,m}} (\mathcal{F}f)(y) \exp(-\operatorname{tr}(\pi y s^{-1} y')) dy = \int_{\mathfrak{M}_{n,m}} f(x) e_s(x) dx.$$

Now we multiply (4.15) by $|s|^{(n-\alpha)/2-d} \exp(-\operatorname{tr}(\varepsilon \pi s^{-1}))$, $d = (m+1)/2$, then integrate in s , and change the order of integration. We obtain

$$\int_{\mathfrak{M}_{n,m}} (\mathcal{F}f)(y) a(y) dy = \int_{\mathfrak{M}_{n,m}} f(x) b(x) dx,$$

where

$$\begin{aligned} a(y) &= \int_{\mathcal{P}_m} |s|^{-\alpha/2-d} \exp[-\operatorname{tr}(\pi s^{-1}(y'y + \varepsilon I_m))] ds \quad (s = t^{-1}) \\ &= \int_{\mathcal{P}_m} |t|^{\alpha/2-d} \exp[-\operatorname{tr}(\pi t(y'y + \varepsilon I_m))] dt \\ &= \Gamma_m(\alpha/2) \pi^{-\alpha m/2} |y'y + \varepsilon I_m|^{-\alpha/2}, \quad \operatorname{Re} \alpha > m-1, \end{aligned}$$

and

$$\begin{aligned} b(x) &= \int_{\mathcal{P}_m} |s|^{(n-\alpha)/2-d} \exp[-\operatorname{tr}(sx'x/4\pi + \varepsilon \pi s^{-1})] ds \\ &= (\varepsilon \pi)^{m(n-\alpha)/2} \int_{\mathcal{P}_m} |u|^{(n-\alpha)/2-d} \exp[-\operatorname{tr}(u \varepsilon x'x/4 + u^{-1})] du \\ &= (\varepsilon \pi)^{m(n-\alpha)/2} \mathcal{K}_{(\alpha-n)/2} \left(\frac{\varepsilon}{4} x'x \right). \end{aligned}$$

This gives (4.11). Application of the Fubini theorem in this argument is justified because both integrals in (4.11) converge absolutely. Indeed, for the left-hand side the absolute convergence is obvious, and for the right-hand side it follows from (4.14).

(ii) **The case** $2m \geq n+2$. In this case the interval (4.12) is void. To circumvent this difficulty, we replace $f(x)$ in (4.15) by $|x|_m^{2k} \Delta^k f(x)$ and proceed as above, assuming k large enough and $\operatorname{Re} \alpha > m-1$. Formally we obtain

$$\begin{aligned} (4.16) \quad & \int_{\mathfrak{M}_{n,m}} \Delta^k [|y|_m^{2k} (\mathcal{F}f)(y)] |\varepsilon I_m + y'y|^{-\alpha/2} dy \\ &= \frac{\pi^{nm/2} \varepsilon^{m(n-\alpha)/2}}{\Gamma_m(\alpha/2)} \int_{\mathfrak{M}_{n,m}} |x|_m^{2k} \Delta^k f(x) \mathcal{K}_{(\alpha-n)/2} \left(\frac{\varepsilon}{4} x'x \right) dx. \end{aligned}$$

The equality (4.16) will become meaningful if we justify application of the Fubini theorem and specify a suitable interval for $\operatorname{Re} \alpha$. Clearly, the left-hand side of (4.16) absolutely converges *for all* $\alpha \in \mathbb{C}$, and one should take care of the right-hand side only. By Lemma 2.4 (ii),

the integral on the right-hand side absolutely converges for $\operatorname{Re} \alpha > m - 1 - 2k$ provided $2k > 2m - 2 - n$. Indeed, for small $|x|_m$ we have

$$|x|_m^{2k} \Delta^k f(x) \mathcal{K}_{(\alpha-n)/2} \left(\frac{\varepsilon}{4} x' x \right) = O(|x|_m^{\lambda-n}),$$

where

$$\lambda = \begin{cases} 2k + n, & \text{if } \operatorname{Re} \alpha > n + m - 1, \\ 2k + n + 1 - m - \varepsilon_0, \forall \varepsilon_0 > 0, & \text{if } n - m + 1 \leq \operatorname{Re} \alpha \leq n + m + 1, \\ 2k + \operatorname{Re} \alpha, & \text{if } \operatorname{Re} \alpha < n - m + 1. \end{cases}$$

If $\operatorname{Re} \alpha > m - 1 - 2k$ and ε_0 is small enough ($\varepsilon_0 < 2k - 2m + n + 2$) then $\lambda > m - 1$ and the desired convergence follows. Thus, (4.16) is justified for $\operatorname{Re} \alpha > m - 1$. Since both sides of (4.16) are analytic functions of α in a larger domain $\operatorname{Re} \alpha > m - 1 - 2k$ containing a non-void strip

$$(4.17) \quad m - 1 - 2k < \operatorname{Re} \alpha < n - m + 1,$$

(4.16) also holds in this strip, and one can utilize (4.13) and (4.14) in order to pass to the limit as $\varepsilon \rightarrow 0$. This gives

$$(4.18) \quad \mathcal{Z}(\Delta^k [|y|_m^{2k} (\mathcal{F}f)(y)], -\alpha) = c_\alpha \mathcal{Z}(\Delta^k f, \alpha + 2k - n),$$

$$(4.19) \quad c_\alpha = \pi^{nm/2} 2^{m(n-\alpha)} \Gamma_m \left(\frac{n-\alpha}{2} \right) / \Gamma_m \left(\frac{\alpha}{2} \right).$$

To transform (4.18), we use the equality

$$(4.20) \quad \Delta^k |x|_m^{\alpha+2k-n} = B_k(\alpha) |x|_m^{\alpha-n},$$

$$B_k(\alpha) = \prod_{i=0}^{m-1} \prod_{j=0}^{k-1} (\alpha - i + 2j)(\alpha - n + 2 + 2j + i) = B_k(n - \alpha - 2k),$$

see (3.7). Then the right-hand side of (4.18) becomes $c_\alpha B_k(\alpha) \mathcal{Z}(f, \alpha - n)$. For the left-hand side we obtain (set $\varphi(y) = |y|_m^{2k} (\mathcal{F}f)(y)$, $\alpha = n - \beta - 2k$)

$$\begin{aligned} \mathcal{Z}(\Delta^k [|y|_m^{2k} (\mathcal{F}f)(y)], -\alpha) &= \mathcal{Z}(\Delta^k \varphi, \beta + 2k - n) \\ &= B_k(\beta) \mathcal{Z}(\varphi, \beta - n) \\ &= B_k(n - \alpha - 2k) \mathcal{Z}(\mathcal{F}f, -\alpha) \\ &= B_k(\alpha) \mathcal{Z}(\mathcal{F}f, -\alpha). \end{aligned}$$

Finally, (4.18) reads

$$\mathcal{Z}(\mathcal{F}f, -\alpha) = c_\alpha \mathcal{Z}(f, \alpha - n), \quad m - 1 - 2k < \operatorname{Re} \alpha < n - m + 1,$$

and the desired equality (4.10) follows by analytic continuation. \square

4.3. Normalized zeta distributions of integral order. It is convenient to introduce a special notation for the normalized zeta integral $\mathcal{Z}(f, \alpha - n)/\Gamma_m(\alpha/2)$ which is an entire function of α . We denote

$$(4.21) \quad \zeta_\alpha(x) = \frac{|x|_m^{\alpha-n}}{\Gamma_m(\alpha/2)}, \quad (\zeta_\alpha, f) = a.c. \frac{1}{\Gamma_m(\alpha/2)} \int_{\mathfrak{M}_{n,m}} f(x) |x|_m^{\alpha-n} dx,$$

where “a.c.” abbreviates analytic continuation. We call ζ_α a *normalized zeta distribution of order α* .

In view of forthcoming applications in Section 6, normalized zeta distributions of integral order deserve special treatment. A striking feature of the distribution ζ_α is that for $\alpha = k$, $k = 0, 1, 2, \dots, m-1$, (outside of the domain of absolute convergence !) it is a positive measure. This measure is supported by a lower-dimensional manifold (in the rank-one case $m = 1$ we have only one point $\alpha = 0$ corresponding to the delta function at the origin).

Definition 4.4. The set

$$(4.22) \quad \mathcal{W} = \{0, 1, 2, \dots, m-1\} \cup \{\alpha : \operatorname{Re} \alpha > m-1\}$$

will be called a *Wallah set* of the normalized zeta distribution ζ_α .

This notion was introduced in [FK] for Riesz distributions associated to symmetric cones. Below we obtain explicit representations of ζ_α for integral values of α (specifically, for $0 < \alpha \leq n$) in the Wallah set (4.22).

Theorem 4.5. Let $f \in \mathcal{S}(\mathfrak{M}_{n,m})$. For $\alpha = k$, $k = 1, 2, \dots, n$,

$$(4.23) \quad (\zeta_k, f) = \frac{\pi^{(n-k)m/2}}{\Gamma_m(n/2)} \int_{SO(n)} d\gamma \int_{\mathfrak{M}_{k,m}} f\left(\gamma \begin{bmatrix} \omega \\ 0 \end{bmatrix}\right) d\omega.$$

Furthermore, in the case $\alpha = 0$ we have

$$(4.24) \quad (\zeta_0, f) = \frac{\pi^{nm/2}}{\Gamma_m(n/2)} f(0).$$

Proof. STEP 1. Let first $k > m-1$. In polar coordinates we have

$$\begin{aligned} \mathcal{Z}(f, k-n) &= \int_{\mathfrak{M}_{n,m}} f(x) |x|_m^{k-n} dx \\ &= 2^{-m} \sigma_{n,m} \int_{\mathcal{P}_m} |r|^{k/2-d} dr \int_{SO(n)} f\left(\gamma \begin{bmatrix} r^{1/2} \\ 0 \end{bmatrix}\right) d\gamma. \end{aligned}$$

Now we replace γ by $\gamma \begin{bmatrix} \beta & 0 \\ 0 & I_{n-k} \end{bmatrix}$, $\beta \in SO(k)$, then integrate in $\beta \in SO(k)$, and replace the integration over $SO(k)$ by that over $V_{k,m}$. We get

$$\begin{aligned} \mathcal{Z}(f, k-n) &= \frac{2^{-m} \sigma_{n,m}}{\sigma_{k,m}} \int_{SO(n)} d\gamma \int_{\mathcal{P}_m} |r|^{k/2-d} dr \int_{V_{k,m}} f \left(\gamma \begin{bmatrix} vr^{1/2} \\ 0 \end{bmatrix} \right) dv \\ &\quad (\text{set } \omega = vr^{1/2} \in \mathfrak{M}_{k,m}) \\ &= \frac{\sigma_{n,m}}{\sigma_{k,m}} \int_{\mathfrak{M}_{k,m}} d\omega \int_{SO(n)} f \left(\gamma \begin{bmatrix} \omega \\ 0 \end{bmatrix} \right) d\gamma. \end{aligned}$$

This coincides with (4.23).

STEP 2. Our next task is to prove that analytic continuation of (ζ_α, f) at the point $\alpha = k$ ($\leq m-1$) has the form (4.23). We first note that for $\alpha = 0$, (4.24) is an immediate consequence of (4.10). Let $k > 0$. We split $x \in \mathfrak{M}_{n,m}$ in two blocks $x = [y; b]$ where $y \in \mathfrak{M}_{n,k}$ and $b \in \mathfrak{M}_{n,m-k}$. Then for $\operatorname{Re} \alpha > m-1$,

$$(\zeta_\alpha, f) = \frac{1}{\Gamma_m(\alpha/2)} \int_{\mathfrak{M}_{n,k}} dy \int_{\mathfrak{M}_{n,m-k}} f([y; b]) \left| \begin{array}{cc} y'y & y'b \\ b'y & b'b \end{array} \right|^{(\alpha-n)/2} db$$

where $\left| \begin{array}{cc} * & * \\ * & * \end{array} \right|$ denotes the determinant of the respective matrix $\begin{bmatrix} * & * \\ * & * \end{bmatrix}$.

By passing to polar coordinates (see Lemma 2.6) $y = vr^{1/2}$, $v \in V_{n,k}$, $r \in \mathcal{P}_k$, we have

$$\begin{aligned} (\zeta_\alpha, f) &= \frac{2^{-k}}{\Gamma_m(\alpha/2)} \int_{V_{n,k}} dv \int_{\mathcal{P}_k} |r|^{(n-k-1)/2} dr \\ &\quad \times \int_{\mathfrak{M}_{n,m-k}} f([vr^{1/2}; b]) \left| \begin{array}{cc} r & r^{1/2}v'b \\ b'vr^{1/2} & b'b \end{array} \right|^{(\alpha-n)/2} db \\ &= \frac{2^{-k} \sigma_{n,k}}{\Gamma_m(\alpha/2)} \int_{SO(n)} d\gamma \int_{\mathcal{P}_k} |r|^{(n-k-1)/2} dr \\ &\quad \times \int_{\mathfrak{M}_{n,m-k}} f_\gamma([\lambda_0 r^{1/2}; b]) \left| \begin{array}{cc} r & r^{1/2} \lambda'_0 b \\ b' \lambda_0 r^{1/2} & b'b \end{array} \right|^{(\alpha-n)/2} db. \end{aligned}$$

Here

$$\lambda_0 = \begin{bmatrix} I_k \\ 0 \end{bmatrix} \in V_{n,k}, \quad f_\gamma(x) = f(\gamma x).$$

We write

$$b = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}, \quad b_1 \in \mathfrak{M}_{k,m-k}, \quad b_2 \in \mathfrak{M}_{n-k,m-k}.$$

Since $\lambda'_0 b = b_1$, then

$$\begin{aligned} (\zeta_\alpha, f) &= \frac{2^{-k} \sigma_{n,k}}{\Gamma_m(\alpha/2)} \int_{SO(n)} d\gamma \int_{\mathcal{P}_k} |r|^{(n-k-1)/2} dr \int_{\mathfrak{M}_{k,m-k}} db_1 \\ &\times \int_{\mathfrak{M}_{n-k,m-k}} f_\gamma \left(\begin{bmatrix} r^{1/2} & b_1 \\ 0 & b_2 \end{bmatrix} \right) \left| \begin{bmatrix} r & r^{1/2} b_1 \\ b'_1 r^{1/2} & b'_1 b_1 + b'_2 b_2 \end{bmatrix} \right|^{(\alpha-n)/2} db_2. \end{aligned}$$

Note that

$$\begin{bmatrix} r & r^{1/2} b_1 \\ b'_1 r^{1/2} & b'_1 b_1 + b'_2 b_2 \end{bmatrix} = \begin{bmatrix} r & 0 \\ b'_1 r^{1/2} & I_{m-k} \end{bmatrix} \begin{bmatrix} I_k & r^{-1/2} b_1 \\ 0 & b'_2 b_2 \end{bmatrix},$$

and

$$\det \begin{bmatrix} r & r^{1/2} b_1 \\ b'_1 r^{1/2} & b'_1 b_1 + b'_2 b_2 \end{bmatrix} = \det(r) \det(b'_2 b_2),$$

see, e.g., [Mu, p. 577]. Therefore,

$$(4.25) \quad (\zeta_\alpha, f) = c_\alpha \int_{SO(n)} d\gamma \int_{\mathcal{P}_k} |r|^{(\alpha-k-1)/2} dr \int_{\mathfrak{M}_{k,m-k}} \psi_{\alpha-k}(\gamma, r, b_1) db_1,$$

where

$$\begin{aligned} c_\alpha &= \frac{2^{-k} \sigma_{n,k} \Gamma_{m-k}((\alpha-k)/2)}{\Gamma_m(\alpha/2)}, \\ \psi_{\alpha-k}(\gamma, r, b_1) &= \frac{1}{\Gamma_{m-k}((\alpha-k)/2)} \int_{\mathfrak{M}_{n-k,m-k}} f_\gamma \left(\begin{bmatrix} r^{1/2} & b_1 \\ 0 & b_2 \end{bmatrix} \right) \\ &\times |b'_2 b_2|^{(\alpha-k)/2 - (n-k)/2} db_2. \end{aligned}$$

The last expression represents the normalized zeta distribution of order $\alpha-k$ in the b_2 -variable. Owing to (4.24), analytic continuation of (4.25) at $\alpha = k$ reads

$$(\zeta_k, f) = c_k \int_{SO(n)} d\gamma \int_{\mathcal{P}_k} |r|^{-1/2} dr \int_{\mathfrak{M}_{k,m-k}} f_\gamma \left(\begin{bmatrix} r^{1/2} & b_1 \\ 0 & 0 \end{bmatrix} \right) db_1,$$

$c_k = [\pi^{(n-k)(m-k)/2} / \Gamma_{m-k}((n-k)/2)] \lim_{\alpha \rightarrow k} c_\alpha$. To transform this expression, we replace γ by $\gamma \begin{bmatrix} \beta & 0 \\ 0 & I_{n-k} \end{bmatrix}$, $\beta \in SO(k)$, and integrate in β . This gives

$$\begin{aligned}
(\zeta_k, f) &= c_k \int_{\mathcal{P}_k} |r|^{-1/2} dr \int_{SO(k)} d\beta \int_{\mathfrak{M}_{k,m-k}} db_1 \\
&\times \int_{SO(n)} f_\gamma \left(\begin{bmatrix} \beta r^{1/2} & \beta b_1 \\ 0 & 0 \end{bmatrix} \right) d\gamma \\
&\quad (\text{set } \zeta = \beta b_1, \quad \eta = b|r|^{1/2} \text{ and use Lemma 2.6}) \\
&= \frac{2^k c_k}{\sigma_{k,k}} \int_{\mathfrak{M}_{k,k}} d\eta \int_{\mathfrak{M}_{k,m-k}} d\zeta \int_{SO(n)} f_\gamma \left(\begin{bmatrix} \eta & \zeta \\ 0 & 0 \end{bmatrix} \right) d\gamma \\
&= c \int_{SO(n)} d\gamma \int_{\mathfrak{M}_{k,m}} f \left(\gamma \begin{bmatrix} \omega \\ 0 \end{bmatrix} \right) d\omega, \\
c &= \frac{\pi^{(n-k)(m-k)/2} \sigma_{n,k}}{\sigma_{k,k} \Gamma_{m-k}((n-k)/2)} \lim_{\alpha \rightarrow k} \frac{\Gamma_{m-k}((\alpha-k)/2)}{\Gamma_m(\alpha/2)} = \frac{\pi^{(n-k)m/2}}{\Gamma_m(n/2)}.
\end{aligned}$$

(here we used formulae (2.8) and (2.31)). \square

The following formulas are consequences of (4.23).

Corollary 4.6. *For all $k = 1, 2, \dots, n$,*

$$(4.26) \quad (\zeta_k, f) = c_1 \int_{V_{n,k}} dv \int_{\mathfrak{M}_{k,m}} f(v\omega) d\omega,$$

$$(4.27) \quad c_1 = 2^{-k} \pi^{(nm-km-nk)/2} \Gamma_k(n/2) / \Gamma_m(n/2).$$

Moreover, if $k = 1, 2, \dots, m-1$, then

$$(4.28) \quad (\zeta_k, f) = c_1 \int_{V_{m,k}} du \int_{\mathfrak{M}_{n,k}} f(yu') |y|_k^{m-n} dy,$$

and

$$(4.29) \quad (\zeta_k, f) = c_2 \int_{\mathfrak{M}_{n,k}} \frac{dy}{|y|_k^{n-m}} \int_{\mathfrak{M}_{k,m-k}} f([y; yz]) dz,$$

$$(4.30) \quad c_2 = \pi^{(m-k)(n/2-k)} / \Gamma_k(k/2) \Gamma_{m-k}((n-k)/2).$$

Proof. From (4.23) we have

$$\begin{aligned} (\zeta_k, f) &= \frac{\pi^{(n-k)m/2}}{\Gamma_m(n/2)} \int_{\mathfrak{M}_{k,m}} d\omega \int_{SO(n)} f(\gamma \lambda_0 \omega) d\gamma \quad \left(\lambda_0 = \begin{bmatrix} I_k \\ 0 \end{bmatrix} \in V_{n,k} \right) \\ &= \frac{\pi^{(n-k)m/2}}{\sigma_{n,k} \Gamma_m(n/2)} \int_{\mathfrak{M}_{k,m}} d\omega \int_{V_{n,k}} f(v\omega) dv \end{aligned}$$

which coincides with (4.26). To prove (4.28), we pass to polar coordinates in (4.26) by setting $\omega' = ur^{1/2}$, $u \in V_{m,k}$, $r \in \mathcal{P}_k$. This gives

$$\begin{aligned} (\zeta_k, f) &= 2^{-k} c_1 \int_{V_{n,k}} dv \int_{\mathcal{P}_k} |r|^{(m-k-1)/2} dr \int_{V_{m,k}} f(vr^{1/2}u') du \\ &= c_1 \int_{V_{m,k}} du \int_{\mathfrak{M}_{n,k}} f(yu') |y|_k^{m-n} dy. \end{aligned}$$

To prove (4.29), we represent ω in (4.26) in the block form $u = [\eta; \zeta]$, $\eta \in \mathfrak{M}_{k,k}$, $\zeta \in \mathfrak{M}_{k,m-k}$, and change the variable $\zeta = \eta z$. This gives

$$(\zeta_k, f) = c_1 \int_{\mathfrak{M}_{k,k}} |\eta|^{m-k} d\eta \int_{\mathfrak{M}_{k,m-k}} dz \int_{V_{n,k}} f(v[\eta; \eta z]) dv.$$

Using Lemma 2.6 repeatedly, and changing variables, we obtain

$$\begin{aligned} (\zeta_k, f) &= 2^{-k} c_1 \sigma_{k,k} \int_{\mathcal{P}_k} |r|^{(m-k-1)/2} dr \int_{\mathfrak{M}_{k,m-k}} dz \int_{V_{n,k}} f(v[r^{1/2}, r^{1/2}z]) dv \\ &= c_1 \sigma_{k,k} \int_{\mathfrak{M}_{n,k}} \frac{dy}{|y|_k^{n-m}} \int_{\mathfrak{M}_{k,m-k}} f([y; yz]) dz. \end{aligned}$$

By (4.27), (2.31) and (2.9), this coincides with (4.29). \square

The representation (4.29) was obtained in [Sh] and [Kh] in a different way. An idea of (4.28) is due to E. Ournucheva.

Remark 4.7. One can also write (ζ_k, f) as

$$(4.31) \quad (\zeta_k, f) = \int_{\mathfrak{M}_{n,m}} f(x) d\nu_k(x), \quad f \in \mathcal{S}(\mathfrak{M}_{n,m}),$$

where ν_k is a positive locally finite measure defined by

$$(4.32) \quad (\nu_k, \varphi) \equiv c_1 \int_{V_{n,k}} dv \int_{\mathfrak{M}_{k,m}} \varphi(v\omega) d\omega, \quad \varphi \in C_c(\mathfrak{M}_{n,m}),$$

$C_c(\mathfrak{M}_{n,m})$ being the space of compactly supported continuous functions on $\mathfrak{M}_{n,m}$; cf. (4.26). In order to characterize the support of ν_k , we denote

$$(4.33) \quad \mathfrak{M}_{n,m}(k) = \{x : x \in \mathfrak{M}_{n,m}, \text{rank}(x) = k\},$$

$$(4.34) \quad \bar{\mathfrak{M}}_{n,m}(k) = \bigcup_{j=0}^k \mathfrak{M}_{n,m}(j) \quad (\text{the closure of } \mathfrak{M}_{n,m}(k)).$$

Lemma 4.8. *The following statements hold.*

(i) $\text{supp } \nu_k = \bar{\mathfrak{M}}_{n,m}(k)$.

(ii) *The manifold $\mathfrak{M}_{n,m}(k)$ is an orbit of $e_k = \begin{bmatrix} I_k & 0 \\ 0 & 0 \end{bmatrix}_{n \times m}$ under the group of transformations*

$$x \rightarrow g_1 x g_2, \quad g_1 \in GL(n, \mathbb{R}), \quad g_2 \in GL(m, \mathbb{R}).$$

(iii) *The manifold $\bar{\mathfrak{M}}_{n,m}(k)$ is a collection of all matrices $x \in \mathfrak{M}_{n,m}$ of the form*

$$(4.35) \quad x = \gamma \begin{bmatrix} \omega \\ 0 \end{bmatrix}, \quad \gamma \in SO(n), \quad \omega \in \mathfrak{M}_{k,m},$$

or

$$(4.36) \quad x = v\omega, \quad v \in V_{n,k}, \quad \omega \in \mathfrak{M}_{k,m}.$$

Proof. (i) Let us consider (4.32). Since $\text{rank}(v\omega) \leq k$, then $(\nu_k, \varphi) = 0$ for all $\varphi \in C_c(\mathfrak{M}_{n,m})$ supported away from $\bar{\mathfrak{M}}_{n,m}(k)$. This means that $\text{supp } \nu_k = \bar{\mathfrak{M}}_{n,m}(k)$.

(ii) Let us show that each $x \in \mathfrak{M}_{n,m}(k)$ is represented in the form $x = g_1 e_k g_2$ for some $g_1 \in GL(n, \mathbb{R})$ and $g_2 \in GL(m, \mathbb{R})$. By Lemma 2.5, each matrix x can be written as $x = u\rho$, where $u \in V_{n,m}$ and $\rho = (x'x)^{1/2}$ is a positive semi-definite $m \times m$ matrix of rank k . By taking

into account that $\rho = g_2' \begin{bmatrix} I_k & 0 \\ 0 & 0 \end{bmatrix}_{m \times m} g_2$ for some $g_2 \in GL(m, \mathbb{R})$, and

$u = \gamma \begin{bmatrix} I_m \\ 0 \end{bmatrix}$ for some $\gamma \in O(n)$, we obtain $x = g_1 e_k g_2$ with

$$g_1 = \gamma \begin{bmatrix} g_2' & 0 \\ 0 & I_{n-m} \end{bmatrix} \in GL(n, \mathbb{R}).$$

(iii) Clearly, each matrix of the form (4.35) or (4.36) has $\text{rank} \leq k$. Conversely, if $\text{rank}(x) \leq k$ then, as above, $x = u\rho = \gamma \begin{bmatrix} \rho \\ 0 \end{bmatrix}$ where

$\gamma \in O(n)$ and $\rho = (x'x)^{1/2}$ is a positive semi-definite $m \times m$ matrix of rank $\leq k$. The latter can be written as

$$\rho = g\lambda g', \quad g \in O(m), \quad \lambda = \text{diag}(\lambda_1, \dots, \lambda_k, 0, \dots, 0),$$

and therefore,

$$x = \gamma \begin{bmatrix} g\lambda \\ 0 \end{bmatrix} g' = \gamma \begin{bmatrix} g & 0 \\ 0 & I_{n-m} \end{bmatrix} \begin{bmatrix} \lambda \\ 0 \end{bmatrix} g' = \gamma_1 \begin{bmatrix} \omega \\ 0 \end{bmatrix}$$

where $\gamma_1 = \gamma \begin{bmatrix} g & 0 \\ 0 & I_{n-m} \end{bmatrix}$ and $\omega \in \mathfrak{M}_{k,m}$. If $\det(\gamma_1) = 1$, we are done. If $\det(\gamma_1) = -1$, one should replace γ_1 by $\gamma_1 e$, $e = \begin{bmatrix} -1 & 0 \\ 0 & I_{n-1} \end{bmatrix}$ and change ω appropriately. The representation (4.36) follows from (4.35). \square

Corollary 4.9. *The integral (4.31) can be written as*

$$(4.37) \quad (\zeta_k, f) = \int_{\text{rank}(x) \leq k} f(x) d\nu_k(x) = \int_{\text{rank}(x)=k} f(x) d\nu_k(x).$$

Indeed, the first equality follows from Lemma 4.8 (i). The second one is clear from the observation that if $\text{rank}(x) \leq k-1$ then by (4.36), $x = v\omega$, $v \in V_{n,k-1}$, $\omega \in \mathfrak{M}_{k-1,m}$. The set of all such pairs (v, ω) has measure 0 in $V_{n,k} \times \mathfrak{M}_{k,m}$.

5. CONVOLUTIONS WITH ZETA DISTRIBUTIONS AND RIESZ POTENTIALS

5.1. Definitions. For α belonging to the Wallah set (4.22), let us consider the convolution operator $\zeta_\alpha*$ defined by

$$(5.1) \quad (\zeta_\alpha * f)(x) = \frac{1}{\Gamma_m(\alpha/2)} \int_{\mathfrak{M}_{n,m}} f(x-y) |y|_m^{\alpha-n} dy$$

if $\text{Re } \alpha > m-1$, and

$$(5.2) \quad (\zeta_k * f)(x) = c_1 \int_{V_{n,k}} dv \int_{\mathfrak{M}_{k,m}} f(x-v\omega) d\omega,$$

$$c_1 = 2^{-k} \pi^{(nm-km-nk)/2} \Gamma_k(n/2) / \Gamma_m(n/2).$$

if $\alpha = k$, $k = 1, 2, \dots, n$; cf. Corollary 4.6. Note that for $m-1 < k \leq n$, both representations are applicable to $\zeta_k * f$.

Another important normalization of the zeta distribution and the corresponding convolutions (5.1) and (5.2) leads to the Riesz distributions and Riesz potentials. We observe that the functional equation (4.10) for the zeta function can be written in the form

$$(5.3) \quad \frac{1}{\gamma_{n,m}(\alpha)} \mathcal{Z}(f, \alpha - n) = (2\pi)^{-nm} \mathcal{Z}(\mathcal{F}f, -\alpha),$$

$$(5.4) \quad \gamma_{n,m}(\alpha) = \frac{2^{\alpha m} \pi^{nm/2} \Gamma_m(\alpha/2)}{\Gamma_m((n-\alpha)/2)}, \quad \alpha \neq n-m+1, n-m+2, \dots$$

Now we have excluded the values $\alpha = n-m+1, n-m+2, \dots$, because these are poles of the gamma function $\Gamma_m((n-\alpha)/2)$ sitting in the numerator in (5.4) (we recall that $m \geq 2$). The corresponding *Riesz distribution* h_α is defined by

$$(5.5) \quad (h_\alpha, f) = \frac{1}{\gamma_{n,m}(\alpha)} \mathcal{Z}(f, \alpha - n) = a.c. \frac{1}{\gamma_{n,m}(\alpha)} \int_{\mathfrak{M}_{n,m}} |x|_m^{\alpha-n} f(x) dx,$$

where $f \in \mathcal{S}(\mathfrak{M}_{n,m})$. For $\operatorname{Re} \alpha > m-1$, the distribution h_α is regular and agrees with the ordinary function $h_\alpha(x) = |x|_m^{\alpha-n} / \gamma_{n,m}(\alpha)$.

The *Riesz potential* of a function $f \in \mathcal{S}(\mathfrak{M}_{n,m})$ is defined by

$$(5.6) \quad (I^\alpha f)(x) = (h_\alpha, f_x) = \frac{1}{\gamma_{n,m}(\alpha)} \mathcal{Z}(f_x, \alpha - n), \quad f_x(\cdot) = f(x - \cdot).$$

For $\operatorname{Re} \alpha > m-1$, $\alpha \neq n-m+1, n-m+2, \dots$, (5.6) is represented in the classical form by the absolutely convergent integral

$$(5.7) \quad (I^\alpha f)(x) = \frac{1}{\gamma_{n,m}(\alpha)} \int_{\mathfrak{M}_{n,m}} f(x-y) |y|_m^{\alpha-n} dy.$$

This integral operator is well known in the rank-one case $m=1$.

Riesz potentials of integral order deserve special mentioning. The following representations are inherited from those for the normalized zeta function; see Theorem 4.5.

Theorem 5.1. *Let $f \in \mathcal{S}(\mathfrak{M}_{n,m})$. Suppose that $\alpha = k$ is a positive integer. If $k \neq n-m+1, n-m+2, \dots$, then*

$$(5.8) \quad (I^k f)(x) = \gamma_1 \int_{SO(n)} d\gamma \int_{\mathfrak{M}_{k,m}} f\left(x - \gamma \begin{bmatrix} \omega \\ 0 \end{bmatrix}\right) d\omega,$$

$$(5.9) \quad = \gamma_2 \int_{V_{n,k}} dv \int_{\mathfrak{M}_{k,m}} f(x - v\omega) d\omega,$$

where

$$(5.10) \quad \gamma_1 = 2^{-km} \pi^{-km/2} \Gamma_m \left(\frac{n-k}{2} \right) / \Gamma_m \left(\frac{n}{2} \right),$$

$$(5.11) \quad \gamma_2 = 2^{-k(m+1)} \pi^{-k(m+n)/2} \Gamma_k \left(\frac{n-m}{2} \right).$$

The constant γ_2 above was evaluated by making use of (2.9).

The following theorem resumes basic properties of Riesz distributions and Riesz potentials.

Theorem 5.2. *Let $f \in \mathcal{S}(\mathfrak{M}_{n,m})$, $\alpha \in \mathbb{C}$, $\alpha \neq n-m+1, n-m+2, \dots$.*

(i) *The Fourier transform of the Riesz distribution h_α is evaluated by the formula $(\mathcal{F}h_\alpha)(y) = |y|_m^{-\alpha}$, the precise meaning of which is*

$$(5.12) \quad (h_\alpha, f) = (2\pi)^{-nm} (|y|_m^{-\alpha}, (\mathcal{F}f)(y)) = (2\pi)^{-nm} \mathcal{Z}(\mathcal{F}f, -\alpha).$$

(ii) *If $k = 0, 1, \dots$, and Δ is the Cayley-Laplace operator, then*

$$(5.13) \quad (-1)^{mk} \Delta^k h_{\alpha+2k} = h_\alpha, \quad \text{i.e.} \quad (-1)^{mk} (h_{\alpha+2k}, \Delta^k f) = (h_\alpha, f),$$

and, therefore,

$$(5.14) \quad (I^{-2k}f)(x) = (-1)^{mk} (\Delta^k f)(x).$$

In particular,

$$(5.15) \quad (I^0 f)(x) = f(x).$$

Proof. (i) follows immediately from the definition (5.5) and the functional equation (5.3). To prove (5.13), for sufficiently large α we have

$$\begin{aligned} \Delta^k h_{\alpha+2k}(x) &= \frac{1}{\gamma_{n,m}(\alpha+2k)} \Delta^k |x|_m^{\alpha+2k-n} \\ &= \frac{B_k(\alpha)}{\gamma_{n,m}(\alpha+2k)} \Delta^k |x|_m^{\alpha-n} \\ &= c h_\alpha(x), \end{aligned}$$

where by (3.8) and (2.7),

$$c = \frac{B_k(\alpha) \gamma_{n,m}(\alpha)}{\gamma_{n,m}(\alpha+2k)} = \frac{B_k(\alpha) \Gamma_m(\alpha/2) \Gamma_m((n-\alpha)/2 - k)}{4^{mk} \Gamma_m(\alpha/2 + k) \Gamma_m((n-\alpha)/2)} = 1.$$

For all admissible $\alpha \in \mathbb{C}$, (5.13) follows by analytic continuation. The equality (5.14) is a consequence of (5.12) and (5.13). Indeed, by (5.6),

$$\begin{aligned}
(I^{-2k}f)(x) &= (h_{-2k}, f_x) \stackrel{(5.13)}{=} (-1)^{mk} (h_0, \Delta^k f_x) \\
&\stackrel{(5.12)}{=} (-1)^{mk} (2\pi)^{-nm} \mathcal{Z}(\mathcal{F}(\Delta^k f_x), 0) \\
&= (-1)^{mk} (\Delta^k f_x)(0) = (-1)^{mk} (\Delta^k f)(x).
\end{aligned}$$

□

5.2. Riesz potentials and heat kernels. It is convenient to study Riesz potentials by making use of the heat kernels and the corresponding Gauss-Weierstrass integrals. In the rank-one case $m = 1$ this approach is described in [Ru1], Section 16. In the higher rank case it was implicitly indicated in [Cl] and [FK]. The key idea is to represent the Riesz potential as a lower-dimensional fractional integral of the corresponding Gauss-Weierstrass integral which is easy to handle.

Definition 5.3. For $x \in \mathfrak{M}_{n,m}$, $n \geq m$, and $t \in \mathcal{P}_m$, we define the (generalized) heat kernel $h_t(x)$ by the formula

$$(5.16) \quad h_t(x) = (4\pi)^{-nm/2} |t|^{-n/2} \exp(-\text{tr}(t^{-1}x'x)/4)$$

where $|t| = \det(t)$. The corresponding Gauss-Weierstrass integral $(W_t f)(x)$ of a function $f(x)$ on $\mathfrak{M}_{n,m}$ is defined by

$$(5.17) \quad (W_t f)(x) = \int_{\mathfrak{M}_{n,m}} h_t(x-y) f(y) dy = \int_{\mathfrak{M}_{n,m}} h_{I_m}(y) f(x-yt^{1/2}) dy.$$

Lemma 5.4.

(i) For each $t \in \mathcal{P}_m$,

$$(5.18) \quad \int_{\mathfrak{M}_{n,m}} h_t(x) dx = 1.$$

(ii) The Fourier transform of $h_t(x)$ has the form

$$(5.19) \quad (\mathcal{F}h_t)(y) = \exp(-\text{tr}(ty'y)),$$

which implies the semi-group property

$$(5.20) \quad h_t * h_\tau = h_{t+\tau}, \quad t, \tau \in \mathcal{P}_m.$$

(iii) If $f \in L^p(\mathfrak{M}_{n,m})$, $1 \leq p \leq \infty$, then

$$(5.21) \quad \|W_t f\|_p \leq \|f\|_p, \quad W_t W_\tau f = W_{t+\tau} f,$$

and

$$(5.22) \quad \lim_{t \rightarrow 0} (W_t f)(x) = f(x)$$

in the L^p -norm. If f is a continuous function vanishing at infinity, then (5.22) holds in the sup-norm.

Proof. To prove (i), we pass to polar coordinates. Owing to (2.5) and (2.31), we obtain

$$\begin{aligned} \int_{\mathfrak{M}_{n,m}} h_t(x) dx &= \frac{2^{-m} \sigma_{n,m}}{(4\pi)^{nm/2} |t|^{n/2}} \int_{P_m} \exp(-\operatorname{tr}(t^{-1}r)/4) |r|^{n/2-d} dr \\ &= \frac{2^{-m} \sigma_{n,m} \Gamma_m(n/2)}{(4\pi)^{nm/2} |t|^{n/2}} |t^{-1}/4|^{-n/2} = 1. \end{aligned}$$

The formula (5.19) is well known (see, e.g., [Herz]), and the proof is elementary. Indeed, by changing variable $x \rightarrow 2xt^{1/2}$ and setting $\eta = 2yt^{1/2}$, we have

$$(\mathcal{F}h_t)(y) = \pi^{-nm/2} \int_{\mathfrak{M}_{n,m}} \exp(-\operatorname{tr}(tx'x)) \exp(i\operatorname{tr}(\eta'x)) dx.$$

This splits into a product of the one-dimensional Fourier transforms of Gaussian functions. The statement (iii) follows from (5.18) and (5.20). Indeed, the relations in (5.21) are clear. Furthermore,

$$(W_t f)(x) - f(x) = \int_{\mathfrak{M}_{n,m}} h_{I_m}(y) [f(x - yt^{1/2}) - f(x)] dy.$$

Hence, by the generalized Minkowski inequality,

$$\|W_t f - f\|_p \leq \int_{\mathfrak{M}_{n,m}} h_{I_m}(y) \|f(\cdot - yt^{1/2}) - f(\cdot)\|_p dy.$$

If t tends to the 0-matrix, then $yt^{1/2} \rightarrow 0$ in \mathbb{R}^{nm} . Since the integrand above does not exceed $2\|f\|_p h_{I_m}(y)$, we can pass to the limit under the sign of integration, and the desired result follows. For continuous functions vanishing at infinity, the argument is similar. \square

Remark 5.5. A challenging open problem is whether (5.22) holds for almost all $x \in \mathfrak{M}_{n,m}$. This fact is well known in the case $m = 1$ [St2], [SW]. It follows from the estimate

$$(5.23) \quad \sup_{t>0} |(W_t f)(x)| \leq (M^* f)(x)$$

where $(M^*f)(x)$ is the Hardy-Littlewood maximal function. It would be desirable to extend this theory to the matrix case when the positive parameter t is replaced by a positive definite matrix.

Let us consider the Gårding-Gindikin fractional integrals on \mathcal{P}_m defined by

$$(5.24) \quad (I_-^\lambda g)(t) = \frac{1}{\Gamma_m(\lambda)} \int_t^\infty g(\tau) |\tau - t|^{\lambda-d} d\tau,$$

where $d = (m+1)/2$, $\operatorname{Re} \lambda > d-1$, and integration is performed over all $\tau \in \mathcal{P}_m$ so that $\tau - t \in \mathcal{P}_m$. The following theorem establishes connection between the Riesz potentials, the Gauss-Weierstrass integrals, and the Gårding-Gindikin fractional integrals.

Theorem 5.6. *Let $m-1 < \operatorname{Re} \alpha < n-m+1$, $d = (m+1)/2$. Then*

$$(5.25) \quad (I^\alpha f)(x) = \frac{1}{\Gamma_m(\alpha/2)} \int_{\mathcal{P}_m} |t|^{\alpha/2-d} (W_t f)(x) dt,$$

$$(5.26) \quad W_t[I^\alpha f](x) = I_-^{\alpha/2}[(W_{(\cdot)} f)(x)](t)$$

provided that integrals on either side of the corresponding equality exist in the Lebesgue sense.

Proof. The right-hand side of (5.25) transforms as follows:

$$\text{r.h.s.} = \frac{(4\pi)^{-nm/2}}{\Gamma_m(\alpha/2)} \int_{\mathcal{P}_m} |t|^{(\alpha-n)/2-d} dt \int_{\mathfrak{M}_{n,m}} f(y) \exp(-\operatorname{tr}(t^{-1}z)) dy$$

where $z = (x-y)'(x-y)/4$. Now we change the order of integration and replace t^{-1} by t . Since

$$\int_{\mathcal{P}_m} |t|^{(n-\alpha)/2-d} \exp(-\operatorname{tr}(tz)) dt = \Gamma_m\left(\frac{n-\alpha}{2}\right) |z|^{(\alpha-n)/2}, \quad |z| \neq 0,$$

(5.25) follows. The validity of (5.26) is a simple consequence of the semi-group property of the Gauss-Weierstrass integral. Indeed,

$$\begin{aligned} W_t[I^\alpha f](x) &= \frac{1}{\Gamma_m(\alpha/2)} \int_{\mathcal{P}_m} |\tau|^{\alpha/2-d} (W_t W_\tau f)(x) d\tau \\ &= \frac{1}{\Gamma_m(\alpha/2)} \int_{\mathcal{P}_m} |\tau|^{\alpha/2-d} (W_{t+\tau} f)(x) d\tau \\ &= I_-^{\alpha/2}[(W_{(\cdot)} f)(x)](t). \end{aligned}$$

□

Theorem 5.6 has a number of remarkable consequences. Firstly, it enables us to invert the Riesz potential by inverting $I_-^{\alpha/2}$ and applying the approximation property $\lim_{t \rightarrow 0} W_t f = f$. We shall study this question in subsequent publications. Secondly, Theorem 5.6 provides a simple proof of the semigroup property of the Riesz potentials under mild assumptions for f .

Corollary 5.7. *Let*

$$(5.27) \quad \operatorname{Re} \alpha > m - 1, \quad \operatorname{Re} \beta > m - 1, \quad \operatorname{Re} (\alpha + \beta) < n - m + 1.$$

Then

$$(5.28) \quad (I^\alpha I^\beta f)(x) = (I^{\alpha+\beta} f)(x)$$

provided that the integral $(I^{\alpha+\beta} f)(x)$ absolutely converges.

Proof. Applying Theorem 5.6 and changing the order of integration, we obtain

$$\begin{aligned} I^\alpha I^\beta f &= \frac{1}{\Gamma_m(\alpha/2)} \int_{\mathcal{P}_m} |t|^{\alpha/2-d} W_t I^\beta f \, dt \\ &= \frac{1}{\Gamma_m(\alpha/2)} \int_{\mathcal{P}_m} |t|^{\alpha/2-d} (I_-^{\beta/2} W_{(\cdot)} f)(t) \, dt \\ &= \frac{1}{\Gamma_m(\alpha/2) \Gamma_m(\beta/2)} \int_{\mathcal{P}_m} W_\tau f \, d\tau \int_0^\tau |t|^{\alpha/2-d} |\tau - t|^{\beta/2-d} \, dt \\ &= \frac{1}{\Gamma_m((\alpha + \beta)/2)} \int_{\mathcal{P}_m} |\tau|^{(\alpha+\beta)/2-d} W_\tau f \, d\tau = I^{\alpha+\beta} f. \end{aligned}$$

□

Remark 5.8. For $m = 1$, the conditions (5.27) have a well-known form

$$\operatorname{Re} \alpha > 0, \quad \operatorname{Re} \beta > 0, \quad \operatorname{Re} (\alpha + \beta) < n.$$

Note that in the higher rank case, (5.27) is possible only if $3m < n + 3$. It is also worth noting that the “usual” way to prove the semigroup property (5.28) for “rough” functions f via the Fourier transform formula and evaluation of the corresponding beta integral (cf. [SKM], [St2]) becomes much more difficult in the higher rank case.

Remark 5.9. Since the “symbol” (i.e., the Fourier transform of the kernel) of the Riesz potential $I^\alpha f$ is $|y|_m^{-\alpha}$, then, for sufficiently good

functions f , (5.28) holds for arbitrary complex α and β . We will be concerned with this topic in Section 6.

5.3. L^p -convolutions. Another useful application of Theorem 5.6 is the following. By Lemma 4.2 and Theorem 4.5, convolutions with zeta distributions and Riesz potentials are well defined on test functions belonging to the Schwartz space. Are they meaningful for $f \in L^p$? Let us study this question.

Theorem 5.10. *For $f \in L^p(\mathfrak{M}_{n,m})$, the Riesz potential $(I^\alpha f)(x)$ absolutely converges almost everywhere on $\mathfrak{M}_{n,m}$ provided*

$$(5.29) \quad \operatorname{Re} \alpha > m - 1, \quad 1 \leq p < \frac{n}{\operatorname{Re} \alpha + m - 1}.$$

Proof. Note that (5.29) is possible if $m \leq [(n+1)/2]$ where $[(n+1)/2]$ is the integral part of $(n+1)/2$. The conditions (5.29) are well known if $m = 1$ [St1] when they have the form $1 \leq p < n/\operatorname{Re} \alpha$. To prove our theorem, it suffices to consider $f \geq 0$ and justify the following inequality

$$(5.30) \quad I \equiv \int_{\mathfrak{M}_{n,m}} \exp(-\operatorname{tr}(x'x)) (I^\alpha f)(x) dx \leq c \|f\|_p, \quad c = \text{const.}$$

Regarding I as the Gauss-Weierstrass integral $(W_{I_m}[I^\alpha f])(0)$ owing to (5.26), we have

$$\begin{aligned} I &\simeq \int_{I_m}^{\infty} |\tau - I_m|^{\alpha/2-d} (W_\tau f)(0) d\tau \\ &\simeq \int_{I_m}^{\infty} |\tau - I_m|^{\alpha/2-d} |\tau|^{-n/2} d\tau \int_{\mathfrak{M}_{n,m}} \exp(-\operatorname{tr}(\tau^{-1}x'x)/4) f(x) dx, \end{aligned}$$

$d = (m+1)/2$. By Hölder's inequality, $I \lesssim A \|f\|_p$ where

$$A^{p'} = \int_{I_m}^{\infty} |\tau - I_m|^{\alpha/2-d} |\tau|^{-n/2} \left(\int_{\mathfrak{M}_{n,m}} \exp(-p' \operatorname{tr}(\tau^{-1}x'x)/4) dx \right)^{1/p'} d\tau,$$

$1/p + 1/p' = 1$. This integral can be easily estimated. Indeed, by setting $x = y\tau^{1/2}$, $dx = |\tau|^{n/2}dy$, we obtain

$$\begin{aligned} A^{p'} &\simeq \int_{I_m}^{\infty} |\tau - I_m|^{\alpha/2-d} |\tau|^{-n/2p} d\tau \quad (\tau^{-1} = r) \\ &\simeq \int_0^{I_m} |I_m - r|^{\alpha/2-d} |r|^{(n/p-\alpha)/2-d} dr. \end{aligned}$$

The last integral is finite if p obeys (5.29). □

More precise information can be obtained for convolutions with zeta distributions (and therefore, for Riesz potentials) of integral order. For $k = 1, 2, \dots, n-1$, we denote

$$(5.31) \quad v_{n-k} = \begin{bmatrix} 0 \\ I_{n-k} \end{bmatrix} \in V_{n,n-k}, \quad \text{Pr}_{\mathbb{R}^{n-k}} = v_{n-k} v'_{n-k} = \begin{bmatrix} 0 & 0 \\ 0 & I_{n-k} \end{bmatrix},$$

$$(5.32) \quad \tilde{f}(x) = \int_{SO(n)} f(\gamma x) d\gamma.$$

Lemma 5.11. *Let $n \geq m \geq 1$; $k = 1, 2, \dots, n$. For any $\lambda > k + m - 1$,*

$$(5.33) \quad \int_{\mathfrak{M}_{n,m}} \frac{|(\zeta_k * f)(x)|}{|I_m + x'x|^{\lambda/2}} dx = c_\lambda \int_{\mathfrak{M}_{n,m}} \frac{\tilde{f}(x)}{|I_m + x' \text{Pr}_{\mathbb{R}^{n-k}} x|^{(\lambda-k)/2}} dx,$$

$$(5.34) \quad c_\lambda = \frac{\pi^{nm/2} \Gamma_m((\lambda - k)/2)}{\Gamma_m(n/2) \Gamma_m(\lambda/2)}.$$

Proof. Suppose first that $k < n$, and denote the left-hand side of (5.33) by $I(f)$. By (5.2),

$$\begin{aligned}
I(f) &= c_1 \int_{V_{n,k}} dv \int_{\mathfrak{M}_{k,m}} d\omega \int_{\mathfrak{M}_{n,m}} f(x - v\omega) \frac{dx}{|I_m + x'x|^{\lambda/2}} \\
&\quad (\text{set } v = \gamma v_0, v_0 = \begin{bmatrix} I_k \\ 0 \end{bmatrix}, x = \gamma y, \quad \gamma \in SO(n)) \\
&= c_1 \sigma_{n,k} \int_{\mathfrak{M}_{k,m}} d\omega \int_{\mathfrak{M}_{n,m}} \tilde{f}(x - v_0\omega) \frac{dx}{|I_m + y'y|^{\lambda/2}} \\
&= c_1 \sigma_{n,k} \int_{\mathfrak{M}_{n-k,m}} db \int_{\mathfrak{M}_{k,m}} \tilde{f}\left(\begin{bmatrix} \omega \\ b \end{bmatrix}\right) d\omega \int_{\mathfrak{M}_{k,m}} \frac{da}{|I_m + b'b + a'a|^{\lambda/2}}.
\end{aligned}$$

For $\lambda > k + m - 1$, the last integral can be explicitly evaluated by the formula

$$(5.35) \quad \int_{\mathfrak{M}_{k,m}} \frac{da}{|q + a'a|^{\lambda/2}} = \frac{\pi^{km/2} \Gamma_m((\lambda - k)/2)}{\Gamma_m(\lambda/2)} |q|^{(k-\lambda)/2}, \quad q \in \mathcal{P}_m$$

(see (A.3)). This gives (5.33). For $k = n$, the proof is simpler and based on (5.35). In this case $\text{Pr}_{\mathbb{R}^{n-k}}$ should be replaced by the zero matrix. \square

Corollary 5.12. *If $n \geq m \geq 1$, $k = 1, 2, \dots, n$, and $\lambda > k + m - 1$, then*

$$(5.36) \quad \int_{\mathfrak{M}_{n,m}} \frac{|(\zeta_k * f)(x)|}{|I_m + x'x|^{\lambda/2}} dx \leq c_\lambda \|f\|_1.$$

If $k = n$ and $f \geq 0$, then (5.36) is a strict equality.

Let us extend (5.36) to $f \in L^p$ with $p \geq 1$. Now we have to impose extra restrictions because of the projection operator $\text{Pr}_{\mathbb{R}^{n-k}}$ on the right-hand side of (5.33).

Theorem 5.13. *Let $n > m$; $k = 1, 2, \dots, n - m$. If $f \in L^p$ and*

$$\lambda > k + \max\left(m - 1, \frac{n + m - 1}{p'}\right), \quad \frac{1}{p} + \frac{1}{p'} = 1,$$

then

$$(5.37) \quad \int_{\mathfrak{M}_{n,m}} \frac{|(\zeta_k * f)(x)|}{|I_m + x'x|^{\lambda/2}} dx \leq c \|f\|_p, \quad c = c(\lambda, p, n, k, m),$$

provided

$$(5.38) \quad 1 \leq p < \frac{n}{k+m-1}.$$

Proof. Note that (5.38) agrees with (5.29) in Theorem 5.10. It suffices to prove the statement for non-negative radial functions $f(x) \equiv f_0(x'x)$. We remind the notation $d = (m+1)/2$. Denote by $I(f)$ the left-hand side of (5.37) and make use of (5.33). Splitting the integral in the right-hand side of (5.33) in $\int_{\mathfrak{M}_{n-k,m}} \times \int_{\mathfrak{M}_{k,m}}$, and passing in $\int_{\mathfrak{M}_{n-k,m}}$ to polar coordinates, we have

$$\begin{aligned} I(f) &\simeq \int_{\mathcal{P}_m} \frac{|r|^{(n-k)/2-d}}{|I_m + r|^{(\lambda-k)/2}} dr \int_{\mathfrak{M}_{k,m}} f_0(\omega'\omega + r) d\omega \\ &= \int_{\mathfrak{M}_{k,m}} d\omega \int_{\omega'\omega}^{\infty} f_0(s) \frac{|s - \omega'\omega|^{(n-k)/2-d}}{|I_m + s - \omega'\omega|^{(\lambda-k)/2}} ds \\ &= \int_{\mathcal{P}_m} f_0(s) ds \int_{\omega'\omega < s} \frac{|s - \omega'\omega|^{(n-k)/2-d}}{|I_m + s - \omega'\omega|^{(\lambda-k)/2}} d\omega \\ &\quad (\text{set } \omega = \eta s^{1/2}, \quad \eta \in \mathfrak{M}_{k,m}, \quad d\omega = |s|^{k/2} d\eta) \\ &= \int_{\mathcal{P}_m} f_0(s) |s|^{n/2-d} ds \int_{\eta'\eta < I_m} \frac{|I_m - \eta'\eta|^{(n-k)/2-d}}{|I_m + s^{1/2}(I_m - \eta'\eta)s^{1/2}|^{(\lambda-k)/2}} d\eta. \end{aligned}$$

By taking into account that

$$(5.39) \quad \|f\|_p \simeq \left(\int_{\mathcal{P}_m} |f_0(s)|^p |s|^{n/2-d} ds \right)^{1/p},$$

owing to the Hölder and the generalized Minkowski inequality, we obtain $I(f) \lesssim c \|f\|_p$ where

$$c = \int_{\eta'\eta < I_m} |I_m - \eta'\eta|^{(n-k)/2-d} A_{\mu}^{1/p'}(I_m - \eta'\eta) d\eta,$$

$$A_{\mu}(r) = \int_{\mathcal{P}_m} \frac{|s|^{n/2-d}}{|I_m + s^{1/2}r s^{1/2}|^{\mu/2}} ds, \quad \mu = (\lambda - k)p', \quad r = I_m - \eta'\eta.$$

Since $|I_m + s^{1/2}rs^{1/2}| = |I_m + r^{1/2}sr^{1/2}|$, by changing variable $s = r^{-1/2}\rho r^{-1/2}$, we have

$$A_\mu(r) = |r|^{-n/2} \int_{\mathcal{P}_m} \frac{|\rho|^{n/2-d}}{|I_m + \rho|^{\mu/2}} d\rho.$$

The last integral is finite for $\mu > n + m - 1$ or $(\lambda - k)p' > n + m - 1$ (moreover, it can be explicitly evaluated by (A.2)). Thus for $\lambda > k + (n + m - 1)/p'$ we have

$$\begin{aligned} c &\simeq \int_{\eta'\eta < I_m} |I_m - \eta'\eta|^{(n-k)/2-n/2p'-d} d\eta \\ &\simeq \int_0^{I_m} |I_m - r|^{(n/p-k)/2-d} |r|^{-1/2} dr < \infty \end{aligned}$$

provided $n/p - k > m - 1$, i.e., $1 \leq p < n/(k + m - 1)$. This completes the proof. \square

Remark 5.14. We do not know whether the conditions (5.38) and (5.29) are sharp. Moreover, it would be interesting to study boundedness properties of the Riesz potential operator I^α on functions $f \in L^p(\mathfrak{M}_{n,m})$ in different norms.

6. RADON TRANSFORMS

6.1. Definitions and basic properties. Let k, n , and m be positive integers, $0 < k < n$, $V_{n,n-k}$ be the Stiefel manifold of orthonormal $(n - k)$ -frames in \mathbb{R}^n . For $\xi \in V_{n,n-k}$, and $t \in \mathfrak{M}_{n-k,m}$, the linear manifold

$$(6.1) \quad \tau = \tau(\xi, t) = \{x \in \mathfrak{M}_{n,m} : \xi'x = t\}$$

will be called a *matrix k -plane* in $\mathfrak{M}_{n,m}$. We denote by \mathfrak{T} the variety of all such planes. For $m = 1$, \mathfrak{T} is an affine Grassmann manifold of k -dimensional planes \mathbb{R}^n . The matrix k -plane Radon transform $\hat{f}(\tau)$ of a function $f(x)$ on $\mathfrak{M}_{n,m}$ assigns to f a collection of integrals of f over all matrix planes $\tau \in \mathfrak{T}$. Namely,

$$\hat{f}(\tau) = \int_{x \in \tau} f(x), \quad \tau \in \mathfrak{T}.$$

The dual Radon transform $\check{\varphi}(x)$ is a mean value of a function $\varphi(\tau)$ on \mathfrak{T} over all matrix planes τ through x :

$$\check{\varphi}(x) = \int_{\tau \ni x} \varphi(\tau), \quad x \in \mathfrak{M}_{n,m}.$$

Precise meaning of these integrals is the following. Let τ be the plane (6.1), $\xi_0 = \begin{bmatrix} 0 \\ I_{n-k} \end{bmatrix} \in V_{n,n-k}$, and $g_\xi \in SO(n)$ be a rotation satisfying $g_\xi \xi_0 = \xi$. Then

$$(6.2) \quad \hat{f}(\tau) \equiv \hat{f}(\xi, t) = \int_{\mathfrak{M}_{k,m}} f \left(g_\xi \begin{bmatrix} \omega \\ t \end{bmatrix} \right) d\omega,$$

and

$$(6.3) \quad \check{\varphi}(x) = \frac{1}{\sigma_{n,n-k}} \int_{V_{n,n-k}} \varphi(\xi, \xi'x) d\xi = \int_{SO(n)} \varphi(\gamma \xi_0, \xi'_0 \gamma'x) d\gamma.$$

Lemma 6.1.

(i) *The duality relation*

$$(6.4) \quad \int_{\mathfrak{M}_{n,m}} f(x) \check{\varphi}(x) dx = \frac{1}{\sigma_{n,n-k}} \int_{V_{n,n-k}} d\xi \int_{\mathfrak{M}_{n-k,m}} \varphi(\xi, t) \hat{f}(\xi, t) dt$$

holds provided that either side of (6.4) is finite for f and φ replaced by $|f|$ and $|\varphi|$, respectively.

(ii) *If $f \in L^1(\mathfrak{M}_{n,m})$, then the Radon transform $\hat{f}(\xi, t)$ exists for all $\xi \in V_{n,n-k}$ and almost all $t \in \mathfrak{M}_{n-k,m}$. Furthermore,*

$$(6.5) \quad \int_{\mathfrak{M}_{n-k,m}} \hat{f}(\xi, t) dt = \int_{\mathfrak{M}_{n,m}} f(x) dx, \quad \forall \xi \in V_{n,n-k}.$$

Proof. By (6.2), the right-hand side of (6.4) has the form

$$(6.6) \quad \frac{1}{\sigma_{n,n-k}} \int_{V_{n,n-k}} d\xi \int_{\mathfrak{M}_{n-k,m}} \varphi(\xi, t) dt \int_{\mathfrak{M}_{k,m}} f \left(g_\xi \begin{bmatrix} \omega \\ t \end{bmatrix} \right) d\omega.$$

Changing variables $x = g_\xi \begin{bmatrix} \omega \\ t \end{bmatrix}$, we have

$$\xi'x = (g_\xi \xi_0)' g_\xi \begin{bmatrix} \omega \\ t \end{bmatrix} = \xi'_0 \begin{bmatrix} \omega \\ t \end{bmatrix} = t.$$

Hence, by the Fubini theorem, (6.6) reads

$$\frac{1}{\sigma_{n,n-k}} \int_{V_{n,n-k}} d\xi \int_{\mathfrak{M}_{n,m}} \varphi(\xi, \xi'x) f(x) dx = \int_{\mathfrak{M}_{n,m}} \check{\varphi}(x) f(x) dx.$$

The statement (ii) is a consequence of the Fubini theorem:

$$\begin{aligned} \int_{\mathfrak{M}_{n-k,m}} \hat{f}(\xi, t) dt &= \int_{\mathfrak{M}_{n-k,m}} dt \int_{\mathfrak{M}_{k,m}} f \left(g_\xi \begin{bmatrix} \omega \\ t \end{bmatrix} \right) d\omega \\ &= \int_{\mathfrak{M}_{n,m}} f(g_\xi x) dx = \int_{\mathfrak{M}_{n,m}} f(x) dx. \end{aligned}$$

□

The following properties of the Radon transform can be easily checked.

Lemma 6.2. *Suppose that the Radon transform*

$$f(x) \longrightarrow \hat{f}(\xi, t), \quad x \in \mathfrak{M}_{n,m}, \quad (\xi, t) \in V_{n,n-k} \times \mathfrak{M}_{n-k,m},$$

exists (at least almost everywhere). Then

$$(6.7) \quad \hat{f}(\xi\theta', \theta t) = \hat{f}(\xi, t), \quad \forall \theta \in O(n-k).$$

If $g(x) = \gamma x \beta + y$ where $\gamma \in O(n)$, $\beta \in O(m)$, $y \in \mathfrak{M}_{n,m}$, then

$$(6.8) \quad (f \circ g)^\wedge(\xi, t) = \hat{f}(\gamma\xi, t\beta + \xi'\gamma'y).$$

In particular, if $f_y(x) = f(x + y)$, then

$$(6.9) \quad \hat{f}_y(\xi, t) = \hat{f}(\xi, \xi'y + t).$$

The equality (6.7) is a matrix analog of the “evenness property” of the classical Radon transform, cf. [Hel].

It is known [OR1] that the Radon transform $f \rightarrow \hat{f}$ is injective on the Schwartz space $\mathcal{S} = \mathcal{S}(\mathfrak{M}_{n,m})$ if and only if $k \leq n - m$. The classical problem is how to reconstruct a function f from the integrals $\hat{f}(\tau)$, $\tau \in \mathfrak{T}$. One of the ways to do this lies via Riesz potentials $I^k f$ of integral order. By Theorem 5.1, for $1 \leq k \leq n - m$ and $f \in \mathcal{S}$ we have

$$(6.10) \quad (I^k f)(x) = \gamma_1 \int_{SO(n)} d\gamma \int_{\mathfrak{M}_{k,m}} f \left(x - \gamma \begin{bmatrix} \omega \\ 0 \end{bmatrix} \right) d\omega,$$

$$(6.11) \quad \gamma_1 = 2^{-km} \pi^{-km/2} \Gamma_m \left(\frac{n-k}{2} \right) / \Gamma_m \left(\frac{n}{2} \right).$$

By Theorem 5.13, this expression is well defined a.e. on $\mathfrak{M}_{n,m}$ as an absolutely convergent integral for any $f \in L^p$ provided $1 \leq p < n/(k+m-1)$. The following important statement establishes connection between Riesz potentials and Radon transforms. It generalizes the classical result of B. Fuglede; see [Fu], [Hel] for $m = 1$.

Theorem 6.3. *Let $1 \leq k \leq n - m$. Then*

$$(6.12) \quad \gamma_1(\hat{f})^\vee(x) = (I^k f)(x),$$

provided the Riesz potential on the right-hand side absolutely converges, e.g., for $f \in L^p$, $1 \leq p < n/(k+m-1)$.

Proof. Let $f_x(y) = f(x+y)$. Then (6.9) yields $\hat{f}_x(\xi, t) = \hat{f}(\xi, \xi'x + t)$. Owing to (6.3) and (6.2), by changing the order of integration, we have

$$\begin{aligned} (\hat{f})^\vee(x) &= \frac{1}{\sigma_{n,n-k}} \int_{V_{n,n-k}} \hat{f}(\xi, \xi'x) d\xi = \frac{1}{\sigma_{n,n-k}} \int_{V_{n,n-k}} \hat{f}_x(\xi, 0) d\xi \\ &= \int_{SO(n)} d\gamma \int_{\mathfrak{M}_{k,m}} f\left(x + \gamma \begin{bmatrix} \omega \\ 0 \end{bmatrix}\right) d\omega. \end{aligned}$$

This coincides with $\gamma_1^{-1}(I^k f)(x)$. \square

Corollary 6.4. *If $f \in L^p$, $1 \leq p < n/(k+m-1)$, then $\hat{f}(\tau)$ is finite for almost all $\tau \in \mathfrak{T}$.*

Remark 6.5. The condition $1 \leq p < n/(k+m-1)$ in Corollary 6.4 can be replaced by the weaker one, namely,

$$(6.13) \quad 1 \leq p < \frac{n+m-1}{k+m-1};$$

see [OR2]. The proof of this fact relies on completely different ideas related to representation of the Radon transform of radial functions by the Gårding-Gindikin fractional integral (5.24). For $m = 1$, both conditions coincide and cannot be improved. We are not sure that for $m > 1$, (6.13) is sufficient for the existence of the Riesz potential $I^k f$ (in contrast to the case $m = 1$); cf. Remark 5.14.

6.2. The inversion problem. Theorem 6.3 reduces the inversion problem for the Radon transform to that for the Riesz potentials (as in the rank-one case [Hel], [Ru2]). Below we show how the unknown function f can be recovered in the framework of the theory of distributions.

Let us consider the Riesz distribution

$$(6.14) \quad (h_\alpha, f) = a.c. \frac{1}{\gamma_{n,m}(\alpha)} \int_{\mathfrak{M}_{n,m}} |x|^{\alpha-n} f(x) dx,$$

where $f \in \mathcal{S} = \mathcal{S}(\mathfrak{M}_{n,m})$. From the Fourier transform formula

$$(h_\alpha, f) = (2\pi)^{-nm}(|y|_m^{-\alpha}, (\mathcal{F}f)(y)),$$

it is evident that the Schwartz class \mathcal{S} does not suit well enough because it is not invariant under multiplication by $|y|_m^{-\alpha}$. To get around this difficulty, we choose another space of test functions. Let $\Psi = \Psi(\mathfrak{M}_{n,m})$ be the collection of all functions $\psi(y) \in \mathcal{S}(\mathfrak{M}_{n,m})$ vanishing on the manifold

$$(6.15) \quad V = \{y : y \in \mathfrak{M}_{n,m}, \text{rank}(y) < m\} = \{y : y \in \mathfrak{M}_{n,m}, |y'y| = 0\}$$

with all derivatives (the coincidence of both sets in (6.15) is clear because $\text{rank}(y) = \text{rank}(y'y)$). The manifold V is a cone in \mathbb{R}^{nm} with vertex 0. The set Ψ is a closed linear subspace of \mathcal{S} . Therefore, it can be regarded as a linear topological space with the induced topology of \mathcal{S} . Let $\Phi = \Phi(\mathfrak{M}_{n,m})$ be the Fourier image of Ψ . Since the Fourier transform \mathcal{F} is an automorphism of \mathcal{S} (i.e., a topological isomorphism of \mathcal{S} onto itself), then Φ is a closed linear subspace of \mathcal{S} . Having been equipped with the induced topology of \mathcal{S} , the space Φ becomes a linear topological space isomorphic to Ψ under the Fourier transform.

The spaces Φ and Ψ were introduced by V.I. Semyanistyi [Se] in the rank-one case $m = 1$. They have proved to be very useful in integral geometry, fractional calculus, and the theory of function spaces. Further generalizations and applications can be found in [Li], [Ru1], [Sa], [SKM].

In our case the following characterization of the space Φ is a consequence of a more general result due to S.G. Samko [Sa].

Theorem 6.6. *The Schwartz function $\phi(x)$ on $\mathfrak{M}_{n,m}$ belongs to the space Φ if and only if it is orthogonal to all polynomials $p(x)$ on any hyperplane τ in \mathbb{R}^{nm} having the form $\tau = \{x : \text{tr}(a'x) = c\}$, $a \in V$:*

$$(6.16) \quad \int_{\tau} p(x)\phi(x)d\mu(x) = 0,$$

$d\mu(x)$ being the induced Lebesgue measure on τ .

Note that for $m = 1$, the space Φ consists of Schwartz functions which are orthogonal to all polynomials on \mathbb{R}^n .

We denote by Φ' the space of all linear continuous functionals (generalized functions) on Φ . It is clear that two \mathcal{S}' -distributions that coincide in the Φ' -sense, differ from each other by an arbitrary \mathcal{S}' -distribution with the Fourier transform supported by V . Since for any complex α , multiplication by $|y|_m^{-\alpha}$ is an automorphism of Ψ , then, according to

the general theory [GSh2], I^α is an automorphism of Φ , and we have

$$(6.17) \quad \mathcal{F}[I^\alpha \phi](y) = |y|_m^{-\alpha} \mathcal{F}[\phi](y), \quad \phi \in \Phi.$$

This gives

$$(6.18) \quad I^\alpha I^\beta \phi = I^{\alpha+\beta} \phi \quad \forall \alpha, \beta \in \mathbb{C}$$

(cf. Remark 5.9), and

$$(6.19) \quad \mathcal{F}[I^\alpha f](y) = |y|_m^{-\alpha} \mathcal{F}[f](y)$$

for all Φ' -distributions f . The last formula implies the following

Theorem 6.7. *The function $f \in L^p(\mathfrak{M}_{n,m})$, $1 \leq p < n/(k+m-1)$, can be recovered from the Radon transform $g = \hat{f}$ in the sense of Φ' -distributions by the formula*

$$(6.20) \quad (f, \phi) = \gamma_1(\check{g}, I^{-k} \phi), \quad \phi \in \Phi,$$

where

$$(I^{-k} \phi)(x) = (\mathcal{F}^{-1} |y|_m^k \mathcal{F} \phi)(x),$$

γ_1 being the constant (6.11).

Proof. We have

$$\begin{aligned} (f, \phi) &= (2\pi)^{-nm} (\mathcal{F}[f], \mathcal{F}[\phi]) \\ &= (2\pi)^{-nm} (|y|_m^{-k} \mathcal{F}[f](y), |y|_m^k \mathcal{F}[\phi](y)) \\ &= (2\pi)^{-nm} ((\mathcal{F}[I^k f])(y), |y|_m^k \mathcal{F}[\phi](y)) \\ &= (2\pi)^{-nm} \gamma_1((\mathcal{F}[\check{g}](y), |y|_m^k \mathcal{F}[\phi](y)) \\ &= \gamma_1(\check{g}, I^{-k} \phi). \end{aligned}$$

□

Remark 6.8. For k even, the Riesz potential $I^k f$ can be inverted (in the sense of Φ' -distributions) by repeated application of the Cayley-Laplace operator $\Delta_m = \det(\partial' \partial)$, $\partial = (\partial/\partial x_{i,j})$. This operator agrees with multiplication by $(-1)^m |y|_m^2$ in the Fourier terms, and therefore, $(-1)^{km} \Delta_m^k I^k f = f$ in the Φ' -sense.

Remark 6.9. It would be desirable to obtain *pointwise inversion formulas* for $I^k f$ and \hat{f} (not in the Φ' -sense). In the rank-one case such formulas can be found in [Ru1] and [Ru2].

7. APPENDIX

The formulas in this section are not new. Since it is not so easy to find simple proofs of them in the literature, we present such proofs for convenience of the reader. We recall that \mathcal{P}_m denotes the cone of positive definite $m \times m$ matrices, $\overline{\mathcal{P}}_m$ is the closure of \mathcal{P}_m , B_m is the beta function (2.10), $d = (m + 1)/2$. The following formulas hold:

$$(A.1) \quad \int_s^\infty |r|^{-\gamma} |r - s|^{\alpha-d} dr = |s|^{\alpha-\gamma} B_m(\alpha, \gamma - \alpha),$$

$$s \in \mathcal{P}_m, \quad \operatorname{Re} \alpha > d - 1, \quad \operatorname{Re} (\gamma - \alpha) > d - 1;$$

$$(A.2) \quad \int_s^\infty |I_m + r|^{-\gamma} |r - s|^{\alpha-d} dr = |I_m + s|^{\alpha-\gamma} B_m(\alpha, \gamma - \alpha),$$

$$s \in \overline{\mathcal{P}}_m, \quad \operatorname{Re} \alpha > d - 1, \quad \operatorname{Re} (\gamma - \alpha) > d - 1;$$

$$(A.3) \quad \int_{\mathfrak{M}_{k,m}} |b + y'y|^{-\lambda/2} dy = \frac{\pi^{km/2} \Gamma_m((\lambda - k)/2)}{\Gamma_m(\lambda/2)} |b|^{(k-\lambda)/2},$$

$$b \in \mathcal{P}_m, \quad \operatorname{Re} \lambda > k + m - 1;$$

$$(A.4) \quad \int_{\{y \in \mathfrak{M}_{k,m} : y'y < b\}} |b - y'y|^{(\lambda-k)/2-d} dy = \frac{\pi^{km/2} \Gamma_m((\lambda - k)/2)}{\Gamma_m(\lambda/2)} |b|^{\lambda/2-d},$$

$$b \in \mathcal{P}_m, \quad \operatorname{Re} \lambda > k + m - 1.$$

Proof.

(A.1), (A.2). By setting $r = q^{-1}$, $dr = |q|^{-m-1} dq$, one can write the left side of (A.1) as

$$|s|^{\alpha-d} \int_0^{s^{-1}} |q|^{\gamma-\alpha-d} |s^{-1} - q|^{\alpha-d} dq = |s|^{\alpha-\gamma} B_m(\alpha, \gamma - \alpha),$$

and we are done. The equality (A.2) follows from (A.1) if we replace s and r by $I_m + s$ and $I_m + r$, respectively.

(A.3), (A.4). By changing variable $y \rightarrow yb^{1/2}$, we obtain

$$\int_{\mathfrak{M}_{k,m}} |b + y'y|^{-\lambda/2} dy = |b|^{(k-\lambda)/2} J_1,$$

$$\int_{\{y \in \mathfrak{M}_{k,m} : y'y < b\}} |b - y'y|^{(\lambda-k)/2-d} dy = |b|^{\lambda/2-d} J_2,$$

where

$$\begin{aligned} J_1 &= \int_{\mathfrak{M}_{k,m}} |I_m + y'y|^{-\lambda/2} dy, \\ J_2 &= \int_{\{y \in \mathfrak{M}_{k,m} : y'y < I_m\}} |I_m - y'y|^{(\lambda-k)/2-d} dy. \end{aligned}$$

Thus we have to show that

$$J_1 = J_2 = \frac{\pi^{km/2} \Gamma_m((\lambda - k)/2)}{\Gamma_m(\lambda/2)}.$$

The case $k \geq m$. We write both integrals in the polar coordinates according to Lemma 2.6. For J_1 we have

$$\begin{aligned} J_1 &= 2^{-m} \sigma_{k,m} \int_{\mathcal{P}_m} |r|^{k/2-d} |I_m + r|^{-\lambda/2} dr \\ &= 2^{-m} \sigma_{k,m} B_m \left(\frac{k}{2}, \frac{\lambda - k}{2} \right) \end{aligned}$$

(the second equality holds by (A.2) (with $s = 0$, $\alpha = k/2$, $\gamma = \lambda/2$). Similarly,

$$\begin{aligned} J_2 &= 2^{-m} \sigma_{k,m} \int_0^{I_m} |r|^{k/2-d} |I_m - r|^{(\lambda-k)/2-d} dr \\ &= 2^{-m} \sigma_{k,m} B_m \left(\frac{k}{2}, \frac{\lambda - k}{2} \right). \end{aligned}$$

Now the result follows by (2.10) and (2.31).

The case $k < m$. We replace y by y' and pass to the polar coordinates. This yields

$$\begin{aligned} J_1 &= \int_{\mathfrak{M}_{m,k}} |I_m + yy'|^{-\lambda/2} dy \\ &= 2^{-k} \int_{V_{m,k}} dv \int_{\mathcal{P}_k} |I_m + vqv'|^{-\lambda/2} |q|^{(m-k-1)/2} dq, \\ &\quad (|I_m + vqv'| = |I_k + q|) \\ &= 2^{-k} \sigma_{m,k} \int_{\mathcal{P}_k} |q|^{(m-k-1)/2} |I_k + q|^{-\lambda/2} dq. \end{aligned}$$

By (A.2) (with $s = 0$, $m = k$, $\gamma = \lambda/2$) and (2.9),

$$\begin{aligned} J_1 &= 2^{-k} \sigma_{m,k} B_k \left(\frac{m}{2}, \frac{\lambda - m}{2} \right) \\ &= \frac{\pi^{km/2} \Gamma_k((\lambda - m)/2)}{\Gamma_k(\lambda/2)} \\ &= \frac{\pi^{km/2} \Gamma_m((\lambda - k)/2)}{\Gamma_m(\lambda/2)}. \end{aligned}$$

Similarly,

$$\begin{aligned} J_2 &= \int_{\{y \in \mathfrak{M}_{m,k}: yy' < I_m\}} |I_m - yy'|^{(\lambda-k)/2-d} dy \\ &= 2^{-k} \int_{V_{m,k}} dv \int_{\{q \in \mathcal{P}_k: vqv' < I_m\}} |I_m - vqv'|^{(\lambda-k)/2-d} |q|^{(m-k-1)/2} dq \\ &= 2^{-k} \sigma_{m,k} \int_0^{I_k} |I_k - q|^{(\lambda-m-k-1)/2} |q|^{(m-k-1)/2} dq \\ &= 2^{-k} \sigma_{m,k} B_k \left(\frac{m}{2}, \frac{\lambda - m}{2} \right), \end{aligned}$$

and we get the same.

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